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Ordered fields and the axiom of continuity. II

Yoshio TANAKA*

Department of Mathematics

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Abstract

As is well-known, an ordered field is a field which has a linear (total) order and the order topology by this order. Ordered fields have played important roles in the theory of real numbers. As a continuation of [11], we give further characterizations for ordered fields to satisfy the axiom of continuity or Archimedes' axiom. In terms of these axioms, we also observe (algebraic) ordered fields having different topologies from the order topology in detail (as a development of [5] or [13]).

Key words: ordered field, real number field, Archimedes' axiom, axiom of continuity, metrizability, Sorgenfrey topology, Michael topology

Department of Mathematics, Tokyo Gakugei University, 4-1-1 Nukuikita-machi, Koganei-shi, Tokyo 184-8501, Japan

Introduction and preliminaries

Ordered fields have played important roles in the theory of the fields of real numbers, and they give algebraic and topological foundations in *Analysis* (differential and integral calculus), and so on. As a continuation of [11], we give further characterizations for ordered fields to satisfy the axiom of continuity or Archimedes' axiom. In terms of these axioms, we observe ordered fields by different topologies, Sorgenfrey topology and Michael topology (instead of the order topology), as a development of [5] or [13].

Let **R**; Q; and N be respectively the usual real number field; rational number field; and the set of natural numbers. For the definitions of real number fields, see *Addendum*.

In this paper, we assume that all spaces are regular, T_1. Also, we assume that any sequence in spaces is infinite.

Let X be a set lineraly ordered by $\leq$. For $a, b \in X$ with $a < b$, (a, b), $[a, b]$ are defined as in **R**, and let $(a, \infty) = \{x \in X: x > a\}$, $(-\infty, a) = \{x \in X: x < a\}$. A subset I (not single) of X is an *interval* if for any $a, b \in I$ $(a < b)$, $[a, b] \subset I$. A space $(X, \leq)$ is called a *linearly ordered topological space* (or *LOTS*) if X has the subbase $\{(a, \infty), (-\infty, a): a \in X\}$. This topology on X is called the *order topology* (which coincides with the original topology on X). A space is a *generalized ordered space* (or *GO-space*) if it is a subspace of a

* Tokyo Gakugei University (4-1-1 Nukuikita-machi, Koganei-shi, Tokyo, 184-8501, Japan)

LOTS. Every GO-space need not be a LOTS. As is well-known, every GO-space is hereditarily normal. For LOTS or GO-spaces, see [2], [8], or [9], etc.

Let $K = K(+, \times, \leq)$ be a field having a linear order $\leq$ such that for $a, b \in K$, $a < b$ iff (= if and only if) $a + x < b + x$; and for $x > 0$, $a < b$ iff $a \times x < b \times x$. Let us call the field $(K, \leq)$ an (*algebraic*) *ordered field*. Such a field K contains a subfield which is isomorphic to Q, so we assume $Q \subset K$. For $x \in K$, define $|x| \in K$ by $|x| = x$ if $x \geq 0$, and $|x| = -x$ if $x < 0$. Then, for $x, y \in K$, $|x + y| \leq |x| + |y|$. We call $(K, \leq)$ an *ordered field* if K has the order topology by the order $\leq$. Every ordered field is a LOTS having no isolated points.

Let $(K, \leq)$ be an (algebraic) ordered field. A pair $(A|B)$ of non-empty subsets A and B in K is a (Dedekind) *cut* if $K = A \cup B$, $A \cap B = \emptyset$, and for any $x \in A$, $y \in B$, $x < y$. We recall the folowing classical *Archimedes' axiom*, and the *axiom of continuity* which is stronger than Archimedes' axiom.

- *Archimedes' axiom*: For each $\alpha, \beta \in K$ with $0 < \alpha < \beta$, there exists $n \in N$ such that $\beta < n\alpha$ (equivalently, for each $\alpha \in K$, there exists $n \in N$ with $\alpha < n$).
- *Axiom of continuity*: For each cut $(A|B)$ in K, there exists only one of *max* A and *min* B.

An ordered field K is *Archimedean*; *Dedekind-complete* if it satisfies Archimedes' axiom; the Axiom of continuity, respectively. The ordered field Q is Archimedean, but not Dedekind-complete.

Note 1. Let us recall that an ordered field K is Archimedean iff (*) K is order-preseving isomorphic to a subfield F of $\mathbf{R}$ ([3; 0.21], etc.). In (*), K is homeomorphic to the subspace F of the Euclidean line $\mathbf{R}$ in view of [2; 2.7.5(a)]. While, as is well-known, an ordered field K is Dedekind-complete iff K is isomorphic (equivalently, homeomorphic) to $\mathbf{R}$.

As is well-known, a space X is *metrizable* (or *metric*), if X has a *metric* d; that is, a non-negative real valued function d from $X \times X$ into $\mathbf{R}$ satisfying the following conditions (a), (b), (c), and (d).

(a) $d(x, y) = 0$ iff $x = y$.

(b) $d(x, y) = d(y, x)$ (symmmetry).

(c) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

(c) * $d(x, z) \leq Max\ \{d(x, y), d(y, z)\}$.

(d) $G \subset X$ is open in X iff, for each $x \in G$, there exists an $\epsilon \in \mathbf{R}$ $(\epsilon > 0)$ such that G contains $B_d(x; \epsilon)$ $(= \{y \in X: d(x, y) < \epsilon\})$.

A function d is *o-metric* if d satisfies (a) and (d); see [9]. For an *o*-metric d, d is called *symmetric*; *quasi-metric*; *non-archimedean quasi-metric* if d satisfies (b); (c); (c)* (stronger than (c)), respectively. A space (X, d) is *o-metrizable*; *symmetrizable*; *quasi-metrizable*; *non-archimedean quasi-metrizable* if X has the respective function d (which is compatible with the topology in X). Every first countable space is *o*-metrizable. Every quasi-metrizable space (X, d) is first countable ($\{B_d(x; 1/n): n \in N\}$ is a local base at x), but not every symmetrizable space is first countable. Every metrizable space is non-archimedean quasi-metrizable. For these spaces, see [4], [6], [10], or [15], etc.

A space is a *P-space* if any G_δ-set (i.e., the intersection of countably many open subsets) is open (cf. [3]). Every countable set in a *P*-space X is discrete in X.

Note 2. (1) Every ordered field is homogeneous, and hereditarily paracompact ([7]).

(2) (i) Every Archimedean ordered field is separable metrizable (by *Note* 1). For an ordered field K, the following are equivalent ([14]).

(a) K is separable metrizable.

(b) K is Lindelöf.

(c) K is separable.

(ii) For an ordered field K, the following are equivalent.

(a) K is metrizable (equivalently, *o*-metrizable).

(b) K contains a non-discrete countable subspace.

(c) K is not a P-space.

Indeed, for (ii), (a) ⇔ (b); (a) ⇔ (c) is respectively shown in [14]; [7], etc. For the parenthetic part in (a), since $\{0\}$ is not open in K, K contains a sequence converging to 0, hence (b) holds.

(3) Every ordered field need not be Archimedean, not even be metrizable by the following examples (i) and (ii).

(i) For a completely regular space X, let $C(X)$ be the collection of all continuous functions from X to $\mathbf{R}$. For a maximal ideal M of the ring $C(X)$, the residue class field $C(X)/M$ is an ordered field (see [3; 5.4(c)]). In view of Theorems 5.5 and 13.8 in [3], $K = C(X)/M$ is metrizable iff it is homeomorphic to $\mathbf{R}$ (for example, K is not metrizable if $X = \mathbf{R}$, Q, or N. For $X = N$, a direct proof is given in [14; Example 2]).

(ii) Let K be an ordered field. For a set I, let $\mathbf{X} = \{x_i: i \in I\}$ be the set of algebraically independent indeterminates indexed by I. Let $K(\mathbf{X})$ be the field of all rational functions in the variables $x_i \in \mathbf{X}$ with coefficients in K. For example, for $K = Q$ and $\mathbf{X} = \{x\}$, a countable ordered field $Q(x)$ is separable metrizable, non-Archimedean (Q is not dense, but $Q(x) - Q$ is dense in $Q(x)$); see [14; Example 1]). According to [1; Theorem 2.6], for each uncountable cardinal number $\aleph$, there exist one metrizable and one non-metrizable ordered field of cardinality $\aleph$ by some ordered fields $K(\mathbf{X})$. It is shown in [7] that for an ordered field $F = K(\mathbf{X})$, F is metrizable (resp. separable metrizable) iff $\mathbf{X}$ (resp. F) is countable, revising [1].

Results and observations

In the first half of this section, we give or review characterizations for ordered fields (having the order topology) to satisfy Archimedes' axiom or the axiom of continuity. In the latter half, in terms of these axioms, we observe topological properties on (algebraic) ordered fields by two different topologies from the order topology.

A space is *totally disconnected* (or *hereditarily disconnected* ([2])) if any connected subset is a singleton. A space X is *zero-dimensional* if it has a base by open-and-closed sets (i.e., ind $X = 0$). A space X is *strongly zero-dimensional* if Ind $X = 0$; equivalently, dim $X = 0$ under X being normal ([2; Theorem 7.1.10]). For these spaces, see [2], etc. Every completely regular P-space is zero-dimensional, and every zero-dimensional space is completely regular, totally disconnected. The following lemma is shown in [12; Lemma 8] (see [2; 6.3.2(e)] for (a) ⇒ (c) under X being a LOTS).

Lemma 1. For a GO-space X, the following are equivalent.

(a) X is totally disconnected.

(b) X is zero-dimensional.

(c) X is strongly zero-dimensional (equivalently, dim $X = 0$).

Theorem 1. For an ordered field K, (1) and (2) below hold.

(1) The following are equivalent.

(a) K is Archimedean.

(b) $\lim_{n\to\infty} 1/n = 0$ in K (i.e., $\{1/n: n \in N\}$ has a limit point 0 in K).

(c) The set $\{1/n: n \in N\} \cup \{0\}$ is compact in K.

(d) Q is a dense subset of K.

(e) For α, β in K with $\alpha, \beta > 0$, there exist (unique) elements $k \in \omega = \{0\} \cup N$, and $r \in K$ such that $\alpha = k\beta + r$, and $0 \le r < \beta$.

(f) For each $x \in K$, the *Gauss notation* $[x]$ can be defined (i.e., $[x]$ is a (unique) integer k satisfying $k \le x < k + 1$).

(2) The following are equivalent.

(a) K is Dedekind-complete.

(b) Every lower bounded subset has an infinum in K.

(c) K is Archimedean, and every decrearing sequence $\{[a_n, b_n]: n \in N\}$ of closed intervals in K with $\lim_{n\to\infty}(b_n - a_n) = 0$ has only one common point.

(d) Every lower bounded decreasing sequence has a limit point in K.

(e) Every bounded sequence has an accumulation point in K.

(f) K is Archimedean, and every Cauchy sequence $\{a_n: n \in N\}$ (i.e., for each $\epsilon \in K$ ($\epsilon > 0$), there exists $k \in N$ such that $|a_m - a_n| < \epsilon$ if $m, n > k$) has a limit point.

(g) Some (or any) closed interval $[a, b]$ of K is compact.

(h) Every compact subset of K is precisely a bounded and closed set.

(i) Some (or any) closed interval $[a, b]$ of K is connected.

(j) K is connected.

(k) Every connected subset of K is precisely a singleton or an interval.

(l) K is not totally disconnected.

(m) K is not (strongly) zero-dimensional (equivalently, ind $K = 1$ (Ind $K = 1$, or dim $K = 1$)).

Proof. Many of the equivalences (1) and (2) would be well-known or folkloric (see [11], [13], etc.), but let us give proofs for some equivalences.

For (1), let us show that (a), (e), and (f) are equivalent. For (a) ⇒ (e), let $\alpha, \beta > 0$, and $n_0 = min\ \{n \in N: n\beta > \alpha\}$ (the n_0 exists by (a)). Let $k = n_0 - 1 \in \omega$, and $r = \alpha - k\beta$, then $0 \le r < \beta$, thus (e) holds. The uniqueness of k and r in (e) is routinely shown. For (e) ⇒ (f), let $\alpha \in K$. For $\alpha > 0$, put $\beta = 1$ in (e), then for a (unique) $k \in \omega$, $k \le \alpha < k + 1$, thus $[\alpha] = k$. For $\alpha < 0$, $k \le -\alpha < k + 1$, which implies $[\alpha] = -k - 1$. For (f) ⇒ (a), for each $\alpha \in K$ with $\alpha > 0$, $\alpha < [\alpha] + 1 \in N$.

For (2), we show (a), (j), (l), and (m) are equivalent. (l) ⇔ (m) holds by Lemma 1, and (a) ⇔ (j) ⇔ (l) holds by (j) ⇒ (l) ⇒ (a) ⇒ (j), for example. Here, (a) ⇒ (j) is well-know. For (l) ⇒ (a), suppose (a) does not hold. Then, there exists a cut $(A|B)$ in K such that A has no maxima, and B has no minima. Thus A and B are disjoint open sets in K. To see K is totally disconnected, let C be a connected subset of K. Suppose C contains points p, q with $p < q$, then C is not connected (by the obvious homeomorphism f on K such that $f(p) = a, f(q) = b$, where $a \in A$, $b \in B$), a contradiction.

Let K be an ordered field. Let $\omega = \{0\} \cup N \subset K$, and $\mathbf{D} = \{0, 1, 2, \cdots, 9\} \subset \omega$. For $a_0 \in \omega$ and $a_n \in \mathbf{D}$ ($n \in N$), let us call the series having the form $\sum_{n=0}^{\infty} a_n/10^n$ a *decimal* on K, and write this series by a notation $(a_0 \bullet a_1\ a_2 \cdots)$. Let us call a decimal $(a_0 \bullet a_1\ a_2 \cdots)$ a *periodic* (*or recurring*) *decimal* if it has a part of recurring numbers $(a_n, a_{n+1}, \cdots, a_{n+k})$ for some $n, k \in N$. When an increasing sequence $\{\sum_{i=0}^{n} a_i/10^i : n \in N\}$ of (non-zero) finite decimals in K has a limit point $\alpha \in K^+ = \{x \in K: x > 0\}$, let us say that the decimal $\sum_{n=0}^{\infty} a_n/10^n$ *is the point* α. Let us denote this by $\alpha = (a_0 \bullet a_1\ a_2 \cdots)$, and define $-\alpha = -(a_0 \bullet a_1\ a_2 \cdots) < 0$, so we consider only the decimals on K^+, or $Q^+ = \{x \in Q: x > 0\}$.

Lemma 2. (1) For an ordered field K, K is Archimedean iff $\lim_{n\to\infty} 1/10^n = 0$ in K.

(2) For an Archimedean ordered field K, each $\alpha \in Q^+$ is expressed as an infinite periodic decimal in K.

Proof. (1) holds by Theorem 1(1). For (2), let $\alpha = n/m$ ($m, n \in N$). In view of Theorem 1(1), there exist (unique) r_0, $a_0 \in \omega$ with $r_0 = n - ma_0 < m$, and for each $i \in N$, there exist (unique) r_i, $a_i \in \omega$ such that $r_i = 10\ r_{i-1} - ma_i < m$ (hence, $a_i \in \mathbf{D}$). Thus, within m-times, some $r_i = 0$, otherwise the same $r_i > 0$ appears again. The former implies α is finite, so let $\alpha = (a_0 \bullet a_1\ a_2 \cdots a_k\ 000 \cdots)$, but $a_k \neq 0$. Then $\alpha = (a_0 \bullet a_1\ a_2 \cdots (a_k - 1)\ 999 \cdots)$ by (1). For the latter, $(a_0 \bullet a_1\ a_2 \cdots)$ is an infinite periodic decimal, but $|\alpha - (\sum_{i=0}^{n} a_i/10^i)| \le 1/10^n$. Thus, $\alpha = (a_0 \bullet a_1\ a_2 \cdots)$ by (1).

Theorem 2. For an ordered field K, (1) and (2) below hold.

(1) The folowing are equivalent.

(a) K is Archimedean.

(b) Each (or some) $\alpha \in K^+$ is (uniquely) expressed as an infinite decimal in K.

(c) Each (or some) infinite periodic decimal is a point in Q^+.

(d) The set of all infinite periodic decimals is precisely Q^+.

(2) The following are equivalent.

(a) K is Dedekind-complete.

(b) Each infinite decimal is a point in K^+.

(c) Each non-periodic decimal is a point in $K^+ - Q$.

(d) The set of all non-periodic decimals is precisely $K^+ - Q$.

Proof. For (1), for (a) ⇒ (b), if $\alpha \in Q^+$, (b) holds by Lemma 2(2), so let $\alpha \in K^+ - Q$. Let $A = \{x \in Q: x < \alpha\}$, and let $B = Q - A$. We can take $\alpha_0 = max\ \{k \in \omega: k \in A\}$ by (a), and let $a_0 = \alpha_0$. For each $n \in N$, let $a_n = max\ \{k \in \mathbf{D}: \alpha_{n-1} + k/10^n \in A\}$, and let $\alpha_n = \alpha_{n-1} + a_n/10^n$. Then, $a_n \neq 0$ for infinitely many a_n since $\lim_{n\to\infty} 1/10^n = 0$ by Lemma 2(1). For each $n \in N$, $\alpha_n < \alpha < \alpha_n + 1/10^n \in B$, so $0 < \alpha - \alpha_n < 1/10^n$. Thus, $\alpha = (a_0 \bullet a_1\ a_2 \cdots)$ by Lemma 2(1). To show that each $\alpha \in K^+$ is uniquely expressed as an infinite decimal, let $\alpha = a_0.a_1\ a_2 \cdots = b_0.b_1\ b_2 \cdots$. Suppose that there exist a_i, b_i which are first distinct elements, and let $a_i - 1 \geq b_i$. $\alpha = (a_0 \bullet a_1\ a_2 \cdots a_{i-1} a_i \cdots) > (a_0 \bullet a_1\ a_2 \cdots a_i 000 \cdots) = (a_0 \bullet a_1\ a_2 \cdots (a_i - 1)99 \cdots) \geq (b_0 \bullet b_1\ b_2 \cdots b_i 99 \cdots) \geq \alpha = (b_0 \bullet b_1\ b_2 \cdots)$. This is a contradiction. For (a) ⇒ (c), every infinite periodic decimal can be considered as a geometric series with a common ratio $1/10^m$ for some $m \in N$, then (c) holds by Lemma 2(1). For (b) or (c) ⇒ (a), $\lim_{n\to\infty} 1/10^n = 0$, thus (a) holds by Lemma 2(1). (c) ⇔ (d) holds by means of Lemma 2(2).

For (2), (a) ⇒ (b) holds, because each sequence $\{\sum_{i=1}^{n} a_i/10^i : n \in N\}$ ($a_i \in \mathbf{D}$) is increasing and upper bounded, thus it has a limit point by Theorem 1(2). To see (b) ⇔ (c) ⇔ (d), and (b) ⇒ (a), note that (b), (c), or (d) implies that $\lim_{n\to\infty} 1/10^n = 0$, and then K is Archimedean. Then (b) ⇔ (c) ⇔ (d) holds, in view of (1) or Lemma 2(2). For (b) ⇒ (a), suppose (a) does not hold. Then, there exists a cut $(A|B)$ in K, but A has no maxima (assume $0 \in A$), and B has no minima. Then, similarly as in (1), there exist an increasing sequence $L_A = \{\alpha_n: n \in \omega\}$ of finite decimals in A, and a decreasing sequence $L_B = \{\alpha_n + 1/10^n: n \in \omega\}$ in B. Then the sequence L_A has a limit point α by (b). Since $\lim_{n\to\infty} 1/10^n = 0$, the sequence L_B also has the same limit point α. But, A has no maxima and B has no minima, then $\alpha \notin A$ and $\alpha \notin B$, a contradiction.

Let us give characterizations for an ordered field K to be Dedekind-complete by means of continuous functions from closed intervals in K to K. Let K be an ordered field. Let f: $[a, b]$ ($\subset K$) → K. For $c \in [a, b]$, $\lim_{x\to c} f(x) = \alpha$ means that for each $\epsilon > 0$ ($\epsilon \in K$), there exists $\delta > 0$ ($\delta \in K$) such that if $0 < |\ x - c\ | < \delta$ in $[a, b]$, then $|\ f(x) - \alpha\ | < \epsilon$. Thus, the *continuity*; *differentiability*; or *derivative* f' of f would be defined as in the field $\mathbf{R}$. Also, the *integrability* of f would be defined as follows: For $\delta > 0$ and $n \in N$, let $\Delta(\delta) = \{a = x_0, x_1, x_2, \cdots, x_{n-1}, n_n = b\}$ be a subset of $[a, b]$ with $0 < x_i - x_{i-1} < \delta$ ($1 \leq i \leq n$). Take $\zeta_i \in [x_{i-1}, x_i]$ ($1 \leq i \leq n$), and put $\Theta = \{\zeta_i: 1 \leq i \leq n\}$. Let $\Sigma(\Delta(\delta); \Theta) = \sum_{i=1}^{n} f(\zeta_i)(x_i - x_{i-1})$. Then the function f is *integrable* on $[a, b]$ if there exists $s \in K$ such that $\lim_{\delta\to 0} \Sigma(\Delta(\delta); \Theta) = s$ for any sets $\Delta(\delta)$ and Θ. Here, we assume that $\Delta(\delta) \neq \emptyset$ for any $\delta > 0$; equivalently, K is Archimedean. If f is integrable, then f is bounded, but the converse need not be valid even if $K = \mathbf{R}$.

Theorem 3. For an ordered field K, the following (a) ~ (e) are equivalent. We can replace the domain "$[a, b]$" by "$[0, 1]$" in below.

(a) K is Dedekind-complete.

(b) (*Maximum-minimum theorem*) For any continuous function f: $[a, b] \to K$, $f([a, b])$ has a maximum and a minimum (or, a maximum or minimum).

(c) (*Intermediate-value theorem*) For any continuous function f: $[a, b] \to K$, if $f(a) < \alpha < f(b)$, then there exists $c \in (a, b)$ with $f(c) = \alpha$.

(d) (*Mean-value theorem*) For any continuous function f: $[a, b] \to K$, if f is differentiable on (a, b) and $f(a) \leq f(b)$, then there exists $c \in (a, b)$ with $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

(e) K is Archimedean, and for any (bounded) continuous function f: $[a, b] \to K$, f is integrable.

Proof. This theorem would be folkloric (in particular, (a) implies any of (b) ~ (e), as is well-known). The proof is essentially given in [11], but let us review it briefly. To show that one of (b) ~ (e) implies (a), let K be not Dedekind-complete. Then there exists a cut $(A|B)$ such that A has no maxima, and B has no minima. Thus, A and B are open-and-closed sets in K. Let $a \in A$, $b \in B$, and let $A' = [a, b] \cap A$, $B' = [a, b] \cap B$. Define $f(x) = x - a$ (on A'); and $x - b$ (on B'). Also, define $g(x) = 0$ (on A'); and 1 (on B'). Then f, g: $[a, b] \to K$ are continuous; actually, differentiable. But, f does not satisfy (b) nor (d). While, g does not satisfy (c). Also, g is not integrable (indeed, suppose g is integrable. Then there exists $\alpha \in K$ such that for each $\epsilon > 0$, $|\alpha - p|$, $|\alpha - q| < \epsilon$ for some $p \in A$ and $q \in B$. Thus, $\alpha \in A \cap B$ since A and B are closed in K, a contradiction).

Let f: $[a, b] \to K$ with K an ordered field, and $A \subset [a, b]$. Without loss of generalities, let us consider "[0, 1]" instead of "$[a, b]$", and consider "maxima (or upper bounds)" instead of " minima (or lower bounds)" of $f(A)$. In the following theorem, (a) $\Leftrightarrow$ (d) in (2) is shown in [11].

Theorem 4. For an ordered field K, (1) and (2) below hold.
(1) The following are equivalent.
(a) K is Archimedean.
(b) For any continuous function f: $[0, 1] \to K$, and for the sequence $L_0 = \{1/n: n \in N\}$ in $[0, 1]$, $f(L_0 \cup \{0\})$ has a maximum.
(2) The following are equivalent.
(a) K is Dedekind-complete.
(b) For any continuous function f: $[0, 1] \to K$, and for any decreasing sequence L in $[0, 1]$, $f(clL)$ has a maximum.
(c) K is Archimedean, and same as (b), but $f(L)$ has an upper bound.
(d) K is Archimedean, and for any continuous function f: $[0, 1] \to K$, $f([0, 1])$ has an upper bound.

Proof. We show (2) holds since (1) is similarly shown. For (a) $\Rightarrow$ (b) & (d), the sequence L in (b) converges to a point $a \in K$ by Theorem 1(2). Since $f(cl\ L)$ is a convergent sequence, it has a maximum in K. While, by Theorem 3, $f([0, 1])$ has a maximum (hence, upper bound). To see (b) or (c) $\Rightarrow$ (a), suppose that K is not Dedekind-complete. Then, there exists a decreasing sequence $L = \{d_n: n \in N\}$ of $[0, 1]$ having no limit points by Theorem 1(2). Then L is closed discrete. Since the LOTS K is normal, there exists a closed discrete collection $\mathcal{D} = \{[a_n, b_n]: n \in N\}$ in K such that $[a_n, b_n] \subset [0, 1]$, and $a_n < d_n < b_n$. For each $n \in N$, define a continuous function f_n: $[a_n, b_n] \to K$ by $f_n(x) = (n/(d_n - a_n))(x - a_n)$ on $[a_n, d_n]$, and $f_n(x) = (n/(d_n - b_n))(x - b_n)$ on $[d_n, b_n]$. Then, define f: $[0, 1] \to K$ by $f(x) = f_n(x)$ on $[a_n, b_n]$, and $f(x) = 0$ for $x \notin \bigcup_{n=1}^{\infty}(a_n, b_n)$. Since $\mathcal{D}$ is closed discrete in $[0, 1]$, f is continuous on $[0, 1]$. But, $f(clL)$ has no maxima. Also, $f([0, 1])$ has no upper bounds under K being Archimedean.

Remark 1. The author has a question whether it is possible to replace "maximum" by "upper bound" in (b) of (1) and (2) in Theorem 4. When K is first countable, the question is positive (indeed, K has a decreasing sequence $\{\alpha_n: n \in N\}$ converging to 0, so replace "n" by "$1/\alpha_n$" in the definition of f_n there); thus, we can replace "Archimedean" by "first countable" in (c) and (d) in (2). For a real valued continuous function f: $K \to \mathbf{R}$ with K an ordered field, the above question is also positive (by use of a classic *Tietze's extension theorem*).

Now, for an (algebraic) ordered field $(K, \leq)$, we give K the following well-known topologies $\mathcal{T}_S$ and $\mathcal{T}_M$ which are stronger than the order topology.

- $\mathcal{T}_S$ is the topology having a base $\{[a, b): a, b \in K\}$, where $[a, b) = \{x \in K: a \leq x < b\}$.
- $\mathcal{T}_M$ is the order topology, but all points in Q^c $(= K - Q)$ are isolated.

We call $\mathcal{T}_S$ the *Sorgenfrey topology* (or *lower limit topology*), and $\mathcal{T}_M$ the *Michael topology* on the field $\mathbf{R}$. Let us use the same names on the field K as well as $\mathbf{R}$. ($\mathcal{T}_S$ is sometimes defined as the *upper limit topology* having a base $\{(a, b]: a, b \in K\}$. But, let us use the "lower limit topology". These limit topologies are topologically equivalent).

Let us denote the space K having the topology $\mathcal{T}_S$ by K_S, and $\mathcal{T}_M$ by K_M, but denote the space K having the order topology by K (as ever).

Let us say that the space K_S or K_M is *Archimedean*; *Dedekind-complete* if so is the (algebraic) ordered field K respectively.

Note 3. If $K = \mathbf{R}$, then the space K_S (resp. K_M) is called the *Sorgenfrey line* (resp. *Michael line*), and let us denote it by **S** (resp. **M**). These lines have the following basic properties (i), and have important uses (ii) in the study of normality in product spaces. For these see see [2], [9], etc.

(i) **S** and **M** are first countable, but neither of them is metrizable. **S** is hereditarily separable, and hereditarily Lindelöf, hence perfect (i.e., every closed set is a G_δ–set). While, **M** is hereditarily paracompact, but not separable, not Lindelöf, not perfect. Also, neither **S** nor **M** is a LOTS (i.e., there exists no linear order $\leq$ on **S** (resp. **M**) such that $\mathcal{T}_S$ (resp. $\mathcal{T}_M$) coincides with the order topology by $\leq$).

(ii) Neither $\mathbf{S} \times \mathbf{S}$ nor $\mathbf{M} \times (\mathbf{R} - Q)$ is even normal (though they are prodoucts of hereditarily paracompact spaces).

We recall some spaces around developable spaces (or Moore spaces). For a space X, a sequence $\{\mathcal{G}_n : n \in N\}$ of open covers of X is a *w*Δ-*sequence* if, for each $x \in X$, any sequence $\{x_n : n \in N\}$ with $x_n \in St(x, \mathcal{G}_n)$ has a cluster point in X; especially, a *development* if the cluster point is the point x; equivalently, (*) $\{St(x, \mathcal{G}_n) : n \in N\}$ is a local base at x. A space X is a *developable* (resp. *w*Δ-*space*) if it has a development (resp. *w*Δ-sequence), and X is *quasi-developable* if it has a sequence $\{\mathcal{G}_n : n \in N\}$ of collections of open subsets in X satisfying (*). For these, see [2], [4], [8], or [9], etc.

Let us give observations on the spaces K_S and K_M (containing some basic properties on K_S in [13]). For the space $X = K$, K_S, or K_M, let "$A \subset X$" mean that A is a *subspace* of the respective space.

Observation A. (1) K_S and K_M are GO-spaces which are hereditarily paracompact, and (strongly) zero-dimensional.

Indeed, K_S and K_M are GO-spaces in view of [8: (2.2)]. To see K_S is hereditarily paracompact, let $A \subset K_S$, and $\mathcal{V}$ be an open cover of A. Then $\mathcal{V}$ has a refinement $\{[x, a_x) \cap A : x \in A\}$. For $B = \bigcup\{(x, a_x) \cap A : x \in A\}$, $\{[x, a_x) \cap A : x \notin B\}$ is disjoint. But, K is hereditarily meta-compact by Note 2(1), then B is meta-compact. Hence, $\mathcal{V}$ has a point-finite open refinement. Thus A is meta-compact. But, K_S is a GO-space, then hereditarily collectionwise normal, hence A is collectionwise normal. Thus, A is paracompact, as is well-known. Similarly, K_M is hereditarily paracompact. We see K_S and K_M are strongly zero-dimensional. Clearly, K_S is totally disconnected. For K_M, let $A \subset K_M$ be connected. Then $A \subset Q$ is connected. But, the countable subspace Q is zero-dimensional, hence totally disconnected. Then A is a singleton. Since K_S and K_M are totally disconnected GO-spaces, they are strongly zero-dimensional by Lemma 1.

(2) The following are equivalent.

(a) K_S is (hereditarily) separable (resp. K_M is separable).

(b) K_S is (hereditarily) Lindelöf (resp. K_M is countable).

(c) K is separable metrizable (resp. K is countable metrizable).

Indeed, for (a) ⇒ (b), let K_S be separable. To see K_S is hereditarily Lindelöf, let V be open in K_S. Then, V is separable. But, V is paracompact by (1). Then V is Lindelöf. Since any open subset is Lindelöf, K_S is hereditarily Lindelöf. For (b) ⇒ (c), let K_S be Lindelöf, then so is K. Thus, K is separable metrizable by Note 2(2). For (c) ⇒ (a), let $A \subset K_S$. Since K is separable, then so is K_S. Thus K_S is hereditarily Lindelöf, then A has at most countably many isolated points. Then, A is separable, for K is hereditarily separable. For K_M, the equivalence is routinely shown.

(3) (i) Let $X = K_S$, K_M, or K, and let $A \subset X$ be not a disctete subspace. Then (a) ~ (e) below are equivalent (here, in each of (a) ~ (d), we can take any of K_S, K_M, and K as X, or put $A = X$ ($A = K$ in (e))). For $A \subset K_M$, (b) ⇔ (b') A has a σ-disjoint base ⇔ (b") A is quasi-developable (here, we can put $A = X_M$ in (b') or (b")).

(a) $A \subset X$ is (non-archimedean) quasi-metrizable.

(b) $A \subset X$ is first countable (equivalently, *o*-metrizable).

(c) $A \subset X$ contains a non-discrete countable subspace.

(d) $A \subset X$ is not a *P*-space.

(e) $A \subset K$ is metrizable.

Indeed, let us give the proof of the result for case $A = X = K_S$ (the results for other cases would hold in view of the proof). (a) ⇒ (b) ⇒ (c) ⇒ (d) is routine. For (d) ⇒ (e), K_S contains a subset $T = \bigcap \{U_n : n \in N\}$ with U_n open, but T is not open. Then, for some $p \in T$, any neighborhood of P is not contained in T. Thus, each U_n contains some $[p, \alpha_n)$ with $\alpha_{n+1} < \alpha_n$. Then, K_S contains the sequence $\{\alpha_n: n \in N\}$ converging to the point p in K_S, hence in K. Thus, K is metrizable by Note 2(2). For (e) $\Rightarrow$ (a), K is metrizable, then K has a compatible, non-archimedean quasi-metric ρ (see [4] or [6]). Let $d(x, y) = min\ \{\rho(x, y), 1\}$, then (K, d) is non-archimedean quasi-metrizable. Define d_S: $K_S \times K_S \to \mathbf{R}$ by $d_S(x, y) = 1$ if $x > y$, and $d_S(x, y) = d(x, y)$ if $x \leq y$. Then (K_S, d_S) is non-archimedean quasi-metizable (and, so is (K_M, d_M), here $d_M(x, y) = 1$ if $x \in Q^c$ with $x \neq y$, and $d_M(x, y) = d(x, y)$ if $x \in Q$). For (b) $\Rightarrow$ (b') in K_M, K_M has a decreasing sequence $\{\epsilon_n: n \in N\}$ converging to 0. Thus, $\{\{x\}: x \in Q^c\} \cup \{(x - \epsilon_n, x + \epsilon_n): x \in Q, n \in N\}$ is a σ-disjoint base. (b') $\Rightarrow$ (b'') $\Rightarrow$ (b) is obvious.

(ii) Let $A \subset X = K_S$ or K_M be not a discrete subspace. Then (a) and (b) below are equivalent (we can take any of K_S and K_M as X in each of (a), (b)). For $A \subset K_S$, (c) ~ (f) are equivalent. If K_S is Archimedean (or separable), then (a) ~ (f) are equivalent, and (g) holds. For $A \subset K_M$, (c) ~ (g) are equivalent.

(a) $A \subset X$ is separable metrizable.

(b) $A \subset X$ is countable.

(c) A is metrizable.

(d) A is symmetrizable.

(e) A is a $w\Delta$-space.

(f) A is a LOTS which is not a P-space.

(g) A is perfect.

Indeed, let us assume $A \subset K_S$. (a) $\Rightarrow$ (b) is easy, for A has a countable base. (b) $\Rightarrow$ (a) is obvious, and so is (c) $\Rightarrow$ (d) & (e). For (d) or (e) $\Rightarrow$ (c), A is not a discrete space, then for (d), A contains a sequence having a limit; and for (e), A contains an infinite set having an accumulation point by a $w\Delta$-sequence in A. Thus, A is not a P-space, hence it is quasi-metrizable by (i). Thus, for (d), A is developable by [15; Lemma 2]. For (e), let d be a compatible quasi-metric on A, and $\{\mathcal{G}_n: n \in N\}$ be a $w\Delta$-sequence in A. For $x \in A$, $n \in N$, let $G_n(x) = B_d(x; 1/n) \cap St(x, \mathcal{G}_n)$, and $\mathcal{U}_n = \{G_n(x) : x \in A\}$. Then $\{\mathcal{U}_n: n \in N\}$ is a development in A, thus A is also developable. But, A is paracompact by (1). Thus, for (d) or (e), A is metrizable (by [2; 5.4.1]). For (c) $\Rightarrow$ (f), X is totally disconnected GO-space by (1), then so is A. Thus A is strongly zero-dimensional by Lemma 1. Thus, since A is metrizable, A is a LOTS by [2; 6.3.2(f)]. For (f) $\Rightarrow$ (c), since A is not a P-space, A is metrizable in K by (i), thus A has a G_δ-diagonal (i.e., the diagonal $\{(x, x): x \in A\}$ is a G_δ-set in $A \times A$) in K, hence in X. Then A is metrizable, because every LOTS with a G_δ-diagonal is metrizable (see [9; p.460]). In the latter part, for $A \subset K_S$, since K_S is Archimedean (or separable), A is separable, and hereditarily Lindelöf in view of (2) with Note 2(2). Hence, (c) $\Rightarrow$ (a) holds, and (g) holds (by the hereditary Lindelöf property). For $A \subset K_M$, we show that (g) $\Rightarrow$ (c) holds. Since A is perfect, it is not a P-space. Thus, A is quasi-developable by (i). Then A is developable, because every perfect and quasi-developable space is developable (by [4; Theorem 8.6]). Thus, A is metrizable.

Remark 2. In view of Observation A(3), the following hold (cf. Note 3(i)): If K_S is Archimedean (or separable), then any (non-discrete) uncountable $A \subset K_S$ is not even symmetrizable, not a LOTS. But, this result need not hold for $A \subset K_M$ (indeed, note that for $A = Q^c \cup C \subset K_M$ with C compact in K_M, under K_M being Archimedean (or first countable), A is a metrizable LOTS in K_M, because A is perfect; actually, for a closed set F in A, $F \cap C$ is compact in K_M, hence in K. Since K is metrizable, $F \cap C$ is a G_δ-set in K, hence in K_M. Then, F is a union of an open set and a G_δ-set in A. Hence F is a G_δ-set in A). If K_M is Dedekind-complete, then any $A \subset K_M$ containing an interval I in K_M is not even symmetrizable, not a LOTS (indeed, A is not perfect, because $(a, b) \subset I$ is not perfect by the *Baire-catergory theorem* on $(a, b) \subset \mathbf{R}$).

Observation B. (1) (i) For Theorem 1, (1) remains true for K_S and K_M, but delete (d) for K_M. In (2) there, (a), (b), (c) are equivalent for K_S and K_M.

(ii) Related to (2) of Theorem 1, (a), (b), and (c) below are equivalent. For K_M, (a) $\Leftrightarrow$ (c) holds, but in (c) replace "no increasing sequences" by "no sequences converging to a point in Q^c in K".

(a) K_S is Dedekind-complete.

(b) Every lower bounded decreasing sequence in K_S has a limit point.

(c) For $A \subset K_S$, A is compact iff A is a bounded and closed set which contains no increasing sequences.

Indeed, for example, let us show that (a) ⇔ (c) for K_M holds. For (a) ⇒ (c), the "only if" part in (c) is easy, here any compact set in K_M does not contain a sequence L converging to a point in Q^c in K, for L is closed discrete in K_M. For the "if" part in (c), to see the set A is compact (equivalently, countably compact by Observation A(1)) in K_M, we show that any sequence L in A has an accumulation point in A. Since L is bounded in K, L has a subsequence converging to a point p in K in view of Theorem 1(2), thus $p \in Q$. Hence, L has the accumulation point $p \in A$ in K_M. For (c) ⇒ (a), by Theorem 1(2), we show that each lower bounded and decreasing sequence L in K has a limit point in K. So, suppose L has no limit points in K. Then L is bounded and closed in K_M, but L has no sequences with a limit point in Q^c in K. Thus, L is compact in K_M, hence in K. Hence, L has an accumulation (hence, limit) point in K, which is a contradiction.

(2) For $A \subset X = K_S$ or K_M, if A is compact (resp. connected), then A is countable (resp. single) by (3)(ii) (resp. (1)) in Observation A.

Let f: $[a, b]$ ($\subset K_S$) → K_S (let us take $[a, b]$ as a domain $A \subset K_S$). For $c \in [a, b)$, $\lim_{x\to c} f(x) = \alpha$ (in K_S) means that for each $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < x - c < \delta$ in $[a, b]$, then $0 \leq f(x) - \alpha < \epsilon$. Then, f is *continuous at c* if $\alpha = f(c)$ (f is continuous at the isolated point b in $[a, b]$). For f: $[a, b]$ ($\subset K_M$) → K_M, f is *continuous* at $c \in [a, b]$ if $c \in Q^c$; or $c \in Q$ where for each $\epsilon > 0$, there exists $\delta > 0$ such that for $|x - c| < \delta$ in $[a, b]$, $|f(x) - f(c)| < \epsilon$ if $f(c) \in Q$, but $f(x) = f(c)$ if $f(c) \in Q^c$. For f in K_S (or K_M), f is *continuous* if it is continuous at any $c \in [a, b]$ (equivalently, $f^{-1}(G)$ is open in $[a, b]$ for any open set G). As others, for example, f: $[a, b)$ ($\subset K_S$) → K_S is *differentiable* if for each $c \in [a, b)$, there exists α ($= f'(c)$) $\in K_S$ such that $\lim_{h\to 0} (f(c+h) - f(c))/h = \alpha$ in K_S.

Observation C. (1) (i) The Gauss function $y = [x]$: $K_S \to K_S$ (defined in Theorem 1(1) under K being Archimedean) is continuous, but the function is not continuous for K or K_M.

(ii) For a continuous function f: $[a, b]$ ($\subset K$) → K, f: $[a, b]$ ($\subset K_S$) → K_S is continuous if f is increasing (i.e., $f(x) \leq f(y)$ for any $x < y$), and f: $[a, b]$ ($\subset K_M$) → K_M is continuous iff $f^{-1}(y)$ is open in K_M for any $y \in Q^c$. While, a function f: $[a, b]$ ($\subset K_S$) → K_S is not continuous if f is strongly decreasing (i.e., $f(x) > f(y)$ for any $x < y$).

(2) In Theorem 3, *Maximum-minimum theorem* or *Intermediate-value theorem* implies that K_S and K_M are Dedekind-complete (but the converse need not hold in K_S or K_M). Also, for Theorem 4, (1) remains true for K_S and K_M, and (a) ⇔ (b) ⇔ (c) in (2) remains true for K_S, modifying the proof of Theorem 4.

Related to Theorems 3 and 4, we have the following example.

Example: There exists a continuous function f: $[-1, 1]$ ($\subset K_S$) → K_S satisfying (a) or (b) below. Such a function also exists in K_M.

(a) f has no maxima and no minima.

(b) f has no upper nor lower bounds under K_S being Archimedean (or first countable).

Indeed, for K_S (or K_M) being not Dedekind-complete, there exists f satisfying (a) in view of the proof of Theorem 3, so let K_S (or K_M) be Dedekind-complete (hence, first countable). Then it suffices to see the existence of f in (b). For (b), since K_S and K_M are first countable, there exists a decreasing sequence $\{a_n: n \in N\}$ converging to 0 (with $a_1 = 1$). Using these points a_n, we give the following continuous function f satisfying (b).

(i) For $[-1, 1] \subset K_S$, define $f(x) = -1/x$ on $[-1, 0)$, and define $f(1) = 0$, and $f(x) = -1/a_n$ if $1 - a_n \leq x < 1 - a_{n+1}$ on $[0, 1]$. (This is a correction of [5; Figure 5]). (f has no upper bounds without the Assumption in (b), but put $f(x) = 0$ on $[0, 1]$).

(ii) For $[-1, 1] \subset K_M$, take $p \in Q^c$ (if $K_M = Q$, consider K_M as $K_M \subset$ **R**), and we can assume $0 < p < 1/2$. For each $n \in N$, put $d_n = p + 1/n$. Then $D = \{d_n: n \in N\} \subset Q^c$ is closed and discrete in K_M by Theorem 1(1). Define $f(x) = x$ on $[-1, 1]$, but for $d_n \in D$, let $f(\pm d_n) = \pm 1/a_n$ respectively.

(3) For a differentiable function f: $[a, b)$ ($\subset K_S$) → K_S, f is continuous iff $f'(x) \geq 0$ for any $x \in [a, b)$.

In Theorem 3, *Mean-value theorem* for K_S implies that K_S is Dedekind-complete (but the converse need not hold).

For functions from subspaces of K_S (or K_M) to K (or **R**), etc., we could give some (analogous) observations.

(*Addemdum*): As is well-known, definitions of (algebraic) real number fields are given by several manners as is seen (i), (ii), or (iii) below, for example. The real number field R is isomorphic to the usual real number field **R**. In (i) and (ii), we can replace "the axiom of continuity" by any of the equivalent conditions in this paper (if R has the order topology, etc.).

(i) R is defined as an (algebraic) ordered field satisfying the axiom of continuity.

(ii) R is axiomatically defined as a set satisfying the usual four arithmetic operations and order relation, and the axiom of continuity.

(iii) R is defined as the set of cuts $(A|B)$ in the rational number field Q as follows (here, if *min* B exists, move it into A):

For real numbers $\alpha = (A_1|A_2)$ and $\beta = (B_1|B_2)$, define

(a) $\alpha \leq \beta$ iff $A_1 \subset B_1$, where $\alpha = \beta$ iff $A_1 = B_1$.

(b) $\alpha + \beta = (C_1|C_2)$, where $C_1 = \{x + y: x \in A_1, y \in B_1\}$ and $C_2 = Q - C_1$.

(c) For $\alpha, \beta > 0$, $\alpha \times \beta = (C_1|C_2)$, where $C_2 = \{xy: x \in A_2, y \in B_2\}$ and $C_1 = Q - C_2$. Also, define $\alpha \times (-\beta) = -(\alpha \times \beta)$, $(-\alpha) \times \beta = -(\alpha \times \beta)$, $(-\alpha) \times (-\beta) = \alpha \times \beta$, and $\gamma \times 0 = 0 \times \gamma = 0$ for any γ.

The cuts $(C_1|C_2)$ in (b) and (c) are actually cuts in Q by property of the rationals. It is shown that R is Dedekind-complete.

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順 序 体 と 連 続 性 公 理 . Ⅱ

田　中　祥　雄

数学分野

要　旨

周知のごとく，順序体は線形順序（全順序）および（その順序による）順序位相をもつ「体」である。順序体は実数論において重要な役割を演じてきた。論文［1］[1] の続編として, 順序体が「連続性公理」や「アルキメデスの公理」を満たすための特徴付けを新たに与える。これらの公理の観点から，順序位相と異なる位相をもつ（代数的）順序体について，（論文［2］[2] 又は［3］[3] の発展として）詳しく観察する。

キーワード: 順序体，実数体，アルキメデスの公理，連続性公理，距離化可能性，Sorgenfrey 位相，Michael 位相

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