Bulletin of Tokyo Gakugei University, Division of Natural Sciences, **64**: 5 - 13, 2012

Ordered rings and order-preservation

Yoshimi KITAMURA* and Yoshio TANAKA*

Department of Mathematics

(Received for publication; May 25, 2012)

KITAMURA, Y. and TANAKA, Y.: Ordered rings and order-preservation. Bull. Tokyo Gakugei Univ. Div. Nat. Sci., 64: 5-13 (2012)

ISSN 1880-4330

Abstract

We consider ordered rings or ordered fields, and give several related matters and examples. We give characterizations for residue class rings of ordered rings to be ordered rings (or ordered integral domains). Further, we give a characterization for an ordered ring R to satisfy the condition that all homomorphisms of R to any ordered ring are order-preserving.

Key words and phrases: ordered ring, ordered field, residue class ring, order-preserving

Department of Mathematics, Tokyo Gakugei University, 4-1-1 Nukuikita-machi, Koganei-shi, Tokyo 184-8501, Japan

Introduction

The symbol $\mathbb{R}$; $\mathbb{Q}$; $\mathbb{Z}$; or $\mathbb{N}$ is the field of real numbers; the field of rational numbers; the ring of integers; or the set of positive integers, respectively.

The symbol R or R' is a (non-zero) commutative ring, unless otherwise stated.

We recall that R is called an ordered ring (or totally ordered ring) if it has a total order such that this ordering relation is order-preserving with respect to addition and multiplication. In particular, when R is a field, such an R is called an ordered field. (Ordered fields are considered in [3], [4], [5], [6], etc., in terms of the axiom of continuity or Archimedes' axiom).

We consider partially or totally ordered rings, and give several related matters and examples. The ordering, positive or non-negative cones will play important roles in the theory of ordered integral domains or ordered rings. We give a characterization for residue class rings of ordered rings to be ordered rings, or ordered integral domains.

As is well-known, all homomorphisms of $\mathbb{R}$ to any ordered field are order-preserving. However, for some subfield F of $\mathbb{R}$ having different total orders as an ordered field, the identity map on F is not order-preserving. Then, let us consider an ordered field K which satisfies condition: (C) all homomorphisms of K to any ordered field are order-preserving. We note that if K is Archimedean, and the homomorphisms are continuous, then (C) holds (see [6], etc.). We give an example of a non-Archimedean

* Tokyo Gakugei University (4-1-1 Nukuikita-machi, Koganei-shi, Tokyo, 184-8501, Japan)

ordered field satisfying the condition (C). We show that an ordered field K satisfies (C) iff K has only one positive cone; generally, we obtain an analogous result among ordered rings.

Results

First, let us review ordering relations on sets, and partially or totally ordered rings, and related matters.

Let A be a set, and $\leq$ be an ordering relation in A. Then $\leq$ is a *partial order* (or *semi-order*) if it satisfies the following conditions: (i) $x \leq x$ for all $x \in A$, (ii) $x \leq y$ and $y \leq x$ implies $x = y$, and (iii) $x \leq y$ and $y \leq z$ implies $x \leq z$.

We assume that every partially order $\leq$ in A satisfies: for any $x, y \in A$, two of $x < y$, $y < x$, $x = y$ do not hold simultaneously.

A partial order $\leq$ is a *total order* (or *linear order*) if it moreover satisfies the following condition: For any $x, y \in A$, either $x \leq y$ or $y \leq x$.

Definition 1. (1) R is a *partially ordered ring* if the following (pR) holds (cf. [1], etc.).

(pR): R has a partial order $\leq$ satisfying the following conditions:

(or 1) $a \leq b$ implies $a + x \leq b + x$ for all x.

(or 2) $a \leq b$ and $0 \leq x$ implies $ax \leq bx$.

To define such a partial ordering relation on R, it is enough to specify the elements ≥ 0, subject to:

(pR)*: R has a relation $\leq$ satisfying the following conditions:

(or 0)* $a \geq 0$ and $-a \geq 0$ iff $a = 0$.

(or 1)* $a, b \geq 0$ implies $a + b \geq 0$.

(or 2)* $a, b \geq 0$ implies $ab \geq 0$.

(2) As is well-known, R is an *ordered ring* (or *totally ordered ring*) if (R) below holds. When R is an integral domain (resp. field), such an R is called an *ordered integral domain* (resp. *ordered field*).

(R): R has a total order $\leq$ satisfying (or 1) and (or 2).

For a ring R with the order $\leq$ in (R) (or (pR)), we shall denote it by $(R, \leq)$. If $(R, \leq)$ is, for example, an ordered ring, then let us say that R is an *ordered ring by* the order $\leq$.

For $A, B \subset R$, we will use the following notations:

$-A = \{-x \mid x \in A\}$, $A + B = \{x + y \mid x \in A, y \in B\}$, and $A \cdot A = \{xy \mid x, y \in A\}$.

Remark 1. (1) We recall the following (sR) stronger than (R), and (P) related to (sR) (see [2], etc.).

(sR): R has a total order $\leq$, but use "$<$" (instead of "$\leq$") in (or 1) and (or 2).

(P): R has a subset P satisfying the following:

(ord 1) $P \cup (-P) = R \setminus \{0\}$.

(ord 2) $P + P \subset P$, and $P \cdot P \subset P$.

A field R is an ordered field iff R is an ordered ring in the sense of (sR). Every ordered ring R in the sense of (sR) or (P) is an integral domain. But, every ordered ring (in the sense of (R)) need not be an integral domain; see Example 1 below.

(2) Related to (R), we consider the following (P)* weaker than (P).

(P)*: R has a subset S satisfying the following:

(ord 1)* $R = S \cup (-S)$, but $S \cap (-S) = \{0\}$.

(ord 2)* $S + S \subset S$ and $S \cdot S \subset S$.

In (R), put $S = \{x \in R \mid x \geq 0\}$, then S (denoted by $\lambda(\leq)$) satisfies (ord 1)* and (ord 2)* in (P)*. Conversely, in (P)*, define $a \leq b$ if $b - a \in S$ in (P)*, then $\leq$ (denoted by $\mu(S)$) satisfies (or 1) and (or 2) in (R). Also, $\mu(\lambda(\leq)) = \leq$ and $\lambda(\mu(S)) = S$, thus λ and μ are inverse correspondence alternately. The similar holds between (sR) and (P).

A subset P of R satisfying (ord 1) and (ord 2) in (P) is called a *positive cone* of R. Similarly, let us call a subset S of R satisfying (ord 1)* and (ord 2)* in (P)* a *non-negative cone* of R.

In view of Remark 1(2), if R has a non-negative cone (resp. positive cone), then R is an ordered ring (resp. ordered integral domain), and the converse holds.

A map $f: R \to R'$ is a *homomorphism* if $f(x+y) = f(x) + f(y)$, $f(xy) = f(x) f(y)$, and $f(1) = 1'$, where 1; $1'$ is the identity element of R; R', respectively. A homomorphism f is an *isomorphism* (resp. *monomorphism*) if it is bijective (resp. injective). We note that every homomorphism of a field is a monomorphism.

The following example shows that every ordered ring need not be an integral domain.

Example 1. Let $(K, \leq_K)$ be an ordered field. Let R be a vector space over K with a basis $\{1, \alpha\}$. Define multiplication on R as follows: $(a + b\alpha)(c + d\alpha) = ac + (ad + bc)\,\alpha$ $(a, b, c, d \in K)$. Then R is a ring, but not an integral domain since $\alpha^2 = 0$. To make R be an ordered ring, let us show that R has only two extension orders of $\leq_K$, and then R is an ordered ring by these orders.

Proof. For $x = a + b\alpha \neq 0$ in R, define $0 < x$ if $0 <_K a$; or $a = 0$ with $0 <_K b$. We define $x \leq y$ if $0 \leq y - x$. Then $(R, \leq)$ is an ordered ring such that $\leq$ is an extension of $\leq_K$.

To get other extension order $\leq'$ on R, define a map σ of R to itself by $\sigma(a + b\alpha) = a - b\alpha$. Then σ is an isomorphism (with $\sigma^2 = id$). Thus the isomorphism σ induces an order $\leq'$ on R defined by $x \leq' y$ if $\sigma(x) \leq \sigma(y)$. Then $(R, \leq')$ is an ordered ring such that $\leq'$ is an extension of $\leq_K$.

Let $S = \{x \in R \mid 0 \leq x\}$, and $S' = \{x \in R \mid 0 \leq' x\}$. Then $S' = \sigma(S)$.

Now, suppose that $(R, \preceq)$ is an ordered ring such that $\preceq$ is an extension of $\leq_K$. Let $S^* = \{x \in R \mid 0 \preceq x\}$. Let $x = a + b\alpha \in R$ with $x \succ 0$.

Case 1. $\alpha \succ 0$: If $a = 0$, then $0 <_K b$. Thus, $b \succ 0$, so $b\alpha \succ 0$, which implies $x \succ 0$. If $a \neq 0$, then $0 <_K a$, so $a \succ 0$. Noting $x = a\left(1 + \frac{b}{a}\alpha\right) = a\left(1 + \frac{b}{2a}\alpha\right)^2$, we see $x \succ 0$. Hence, $S \subset S^*$, which implies $S = S^*$.

Case 2. $\alpha \prec 0$: Similarly, we have $\sigma(S) = S^*$ as in Case 1.

Consequently, R has only two extension orders (one is $\leq$ defined by S (i.e., $x \leq y$ iff $y - x \in S$), and another is $\leq'$ defined by S' with $S \neq S'$), and R is an ordered ring by these orders.

We assume that any ideal I in R satisfies $I \neq R$.

Let I be an ideal in R. Let R / I be the residue class ring. For $a \in R$, let $[a]$ mean $a + I$ in R / I. Let us recall the following basic definitions in [1] ((1) will play an important role in (2)).

Definition 2. Let $(R, \leq)$ be a partially ordered ring and I an ideal in R.

(1) I is *convex* if whenever $0 \leq x \leq y$ and $y \in I$, then $x \in I$.

(2) For $a \in R$, define $[a] \geq 0$ in R / I if there exists $x \geq 0$ in R with $[a] = [x]$ (with the same symbol $\leq$ in R / I without confusion).

In what follows, we assume that any residue class ring R / I of a partially ordered ring R has the above ordering relation induced from R, unless otherwise stated.

Example 2. Let K be an ordered field. Let $K[x]$ be the polynomial ring over K in one variable x. We define the following total orders $\leq_1$ and $\leq_2$ on $K[x]$: $0 \prec_1 f(x)$ (resp. $0 \prec_2 f(x)$) if the coefficient of the highest (resp. lowest) degree of $f(x)$ is positive. Then $K[x]$ is an ordered ring by these total orders. For a non-zero ideal I in $K[x]$, the following hold.

(1) I is not convex in $(K[x], \leq_1)$.

(2) I is convex in $(K[x], \leq_2)$ iff I is generated by a monomial.

Proof. We recall that $K[x]$ is a principal ideal domain, so I is generated by a polynomial $h(x) = ax^n + \cdots + bx^i$ ($a, b \in K$, $a \neq 0$, $n > i \geq 0$).

For (1), assume $a = 1$ in $h(x)$. Since $0 <_1 1 <_1 h(x) \in I$ and $1 \notin I$, I is not convex in $(K[x], \leq_1)$.

For (2), suppose $h(x)$ is a monomial. Assume $h(x) = x^n$ ($n > 0$). Let $0 <_2 f(x) <_2 g(x) \in I$. Since $g(x) \in I$, there exists $k(x) \in K[x]$ with $g(x) = x^n k(x)$. Further, there exist $q(x)$, $r(x) \in K[x]$ such that $f(x) = x^n q(x) + r(x)$ and $\deg r(x) < n$. Suppose $r(x) \neq 0$. Then the coefficient of the lowest degree of $r(x)$ is positive since $f(x) >_2 0$. Hence $g(x) - f(x) = x^n (k(x) - q(x)) - r(x) <_2 0$, a contradiction. Thus $r(x) = 0$, so $f(x) = x^n q(x) \in I$. Hence I is convex. Conversely, suppose $h(x)$ is not a monomial. Assume $b = 1$ in $h(x)$, and let $f(x) = \frac{1}{2} x^i \in K[x]$. Then $0 <_2 f(x) <_2 h(x) \in I$, but $f(x) \notin I$. Hence I is not convex in $(K[x], \leq_2)$.

Lemma 1 ([1]). Let $(R, \leq)$ be a partially ordered ring, and I be an ideal in R. Then I is convex iff R / I is a partially ordered ring.

Lemma 2. Let $(R, \leq)$ be a partially ordered ring, and I be an ideal in R. Let $S = \{x \in R \mid x \geq 0\}$, and φ be the natural homomorphism of R to R / I defined by $\varphi(a) = [a]$. Then the following (1) and (2) hold.

(1) The following are equivalent.

(a) R / I is an ordered ring.

(b) $\varphi(S)$ is a non-negative cone of R / I.

(c) I is convex, and $R = (S \cup -S) + I$.

(2) The following are equivalent.

(a) R / I is an ordered integral domain.

(b) $\varphi(S \setminus I)$ is a positive cone of R / I.

(c) I is convex and prime, and $R = (S \cup -S) + I$.

Proof. For (1), let $S^* = \varphi(S)$. Note that $\varphi(R) = S^* \cup (-S^*)$ iff $R = (S \cup -S) + I$. Then the implication (a) $\Rightarrow$ (c) $\Rightarrow$ (b) holds by Lemma 1. For (b) $\Rightarrow$ (a), $\varphi(S) = \{[a] \mid [a] \geq 0\}$, then (a) holds.

For (2), note that I is prime iff R / I is an integral domain. (a) $\Rightarrow$ (c) holds by means of (1). For (c) $\Rightarrow$ (b), and (b) $\Rightarrow$ (a) hold by the same way as in (1).

Remark 2. In Lemma 2, for a prime ideal I in R, $R = (S \cup -S) + I$ holds if for each $a \in R \setminus \{0\}$, there exists $b \in R$ with $a^2 = b^2$ and $b > 0$ (indeed, I is prime, then $a - b \in I$ or $a + b \in I$, hence $R = (S \cup -S) + I$).

The following example shows that (i) every R need not be an ordered ring even if R / I is an ordered integral domain, and (ii) the condition $R = (S \cup -S) + I$ is essential in Lemma 2.

Example 3. Let $\mathbb{Z}[x]$ be the polynomial ring over $\mathbb{Z}$ in one variable x. Let I be the ideal in $\mathbb{Z}[x]$ generated by x. Then the following hold.

(i) Let $S = \mathbb{N} \cup \{0\}$. Then $S \cap -S = \{0\}$, $S + S \subset S$, and $S \cdot S \subset S$. Define a partial order $\leq$ by S on $\mathbb{Z}[x]$: $f \leq g$ if $g - f \in S$. Then $\leq$ is not a total order on $\mathbb{Z}[x]$. While, I is convex and prime, and $\mathbb{Z}[x] = (S \cup -S) + I$. Thus, by Lemma 2 (2), $\mathbb{Z}[x] / I$ is an ordered integral domain.

(ii) Let $S' = \{2n \mid n \in \mathbb{N}\} \cup \{0\}$. Replace "$S$" by "$S'$" in (i), and define $\leq'$ by S' as in (i), then the same in (i) holds on S' and $\leq'$. Similarly, I is prime, and convex (thus, $(\mathbb{Z}[x] / I, \leq')$ is a partially ordered ring by Lemma 1). But, $\mathbb{Z}[x] \neq (S' \cup -S') + I$ and $(\mathbb{Z}[x] / I, \leq')$ is not an ordered ring.

For an ordered ring $(R, \leq)$, let $S = \{x \in R \mid x \geq 0\}$. Then $R = S \cup -S$, thus $R = (S \cup -S) + I$. Hence the following theorem holds by Lemma 2.

Theorem 1. Let $(R, \leq)$ be an ordered ring. Then the following (1) and (2) hold under the same notations in Lemma 2.

(1) The following are equivalent.

(a) R / I is an ordered ring.

(b) $\varphi(S)$ is a non-negative cone of R / I.

(c) I is convex.

(2) The following are equivalent.

(a) R / I is an ordered integral domain.

(b) $\varphi(S \setminus I)$ is a positive cone of R / I,

(c) I is convex and prime.

Corollary 1. Let $(R, \leq)$ be an ordered integral domain. Then the following are equivalent under the same notations in Lemma 2, but use $P = \{x \in R \mid x > 0\}$ (instead of S).

(1) R / I is an ordered integral domain.

(2) $\varphi(P) \setminus \{0\}$ is a positive cone of R / I.

(3) For any x, y in $P \setminus I$, $x + y \notin I$ and $xy \notin I$.

(4) I is convex and prime.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4) holds by Theorem 1(2), and (2) $\Rightarrow$ (3) is obvious. For (3) $\Rightarrow$ (1) & (2), let $P^* = \varphi(P) \setminus \{0\}$. By (3), $P^* + P^* \subset P^*$ and $P^* \cdot P^* \subset P^*$. Also, $P^* \cup -P^* = \varphi(R) \setminus \{0\}$. Hence, (1) and (2) hold.

In (4) of Corollary 1, the convexity of I is essential, and so is the primeness. Indeed, we have the following example.

Example 4. (1) An ordered integral domain $(R, \leq)$ having an ideal I which is prime, but not convex. (Here, R / I is not a partially ordered ring by the order $\leq$ in the sense of Definition 2, but it is an ordered field by some total order).

(2) An ordered integral domain R' having an ideal I which is convex, but not prime. (Here, R' / I is an ordered ring).

Proof. For (1), let $R = K[x]$ be the ordered integral domain by the order $\leq = \leq_1$ in Example 2. Let I be the ideal in R generated by x. Then I is prime, but it is not convex in $(R, \leq)$ by Example 2 (1). For the parenthetic part, the first half holds by Lemma 1. For the latter part, R / I is isomorphic to the ordered field K, thus R / I is an ordered field by some total order.

For (2), let $R' = K[x]$ be the same as (1) with $\leq = \leq_2$ in Example 2. Let I be the ideal in R' generated by x^2. Then I is not prime. While, I is convex by Example 2 (2). Thus the parenthetic part holds by Theorem 1 (1).

Now, let us recall the classic ring $C(X) / I$ in [1] as an important case of the residue class ring R / I. For a completely regular space X, let $C(X)$ be the set of all continuous maps from X to $\mathbb{R}$. Then $C(X)$ is a partially ordered ring (that is; for $f, g \in C(X)$, define $(f + g)(x) = f(x) + g(x)$, $fg(x) = f(x)\,g(x)$; and for $r \in \mathbb{R}$, $\mathbf{r} \in C(X)$ is the constant map $\mathbf{r}(x) \equiv r$. Define a partial order $\leq$ on $C(X)$ by $f \leq g$ if $f(x) \leq g(x)$ for all $x \in X$).

A map $f : (R, \leq) \to (R', \leq')$ is called *order-preserving* if for $x \leq y$, $f(x) \leq' f(y)$.

Let us recall the following classic result [1; Theorem 5.5].

For a prime ideal I in $C(X)$, $C(X) / I$ is an ordered ring (with I convex), and the map $\eta : \mathbb{R} \to C(X) / I$ defined by $\eta(r) = [\mathbf{r}]$ is an order-preserving monomorphism.

Let M be a maximal ideal in $C(X)$. In view of the above, the ordered field $C(X)/M$ contains a canonical copy of $\mathbb{R}$ as a subfield. We recall that $C(X)/M$ is *real* if it is isomorphic to $\mathbb{R}$, and *hyper-real* if it is not real ([1; 5.6]).

For an ordered field $(K, \leq)$, K is *Archimedean* if for each $\alpha, \beta \in K$ with $0 < \alpha < \beta$, there exists $n \in \mathbb{N}$ such that $\beta < n\alpha$.

We note that $C(X)/M$ is real (resp. hyper-real) iff it is Archimedean (resp. non-Archimedean).

Remark 3. For a non-pseudocompact space X (i.e., some $f \in C(X)$ is unbounded, such as $\mathbb{N}$, $\mathbb{Q}$, or $\mathbb{R}$, etc.), there exists a maximal ideal M in $C(X)$ such that $C(X)/M$ is hyper-real, but every ordered field $C(X)/M$ need not be hyper-real; see [1; Theorems 5.8 (b) and 5.14]. While, for a Lindelöf space (i.e., every open cover has a countable subcover), more generally a real compact space X, an ordered field $C(X)/M$ is hyper-real if $\cap \{f^{-1}(0) \mid f \in M\} = \emptyset$; see [1; 5.9, 5.15, or 8.2]. Thus, for example, let $X = \mathbb{N}$ (so $C(X) = \mathbb{R}^{\mathbb{N}}$), and $\mathcal{F}$ be an ultra-filter on $\mathbb{N}$ with $\cap \mathcal{F} = \emptyset$. Then $C(\mathbb{N})/M$ is hyper-real for the maximal ideal $M = \{f \in \mathbb{R}^{\mathbb{N}} \mid f^{-1}(0) \in \mathcal{F}\}$.

In view of these, the second author has the following correction:

In [5; Note 2 (3)] and [6; Example 3.3], the statement that the ordered field $C(\mathbb{N})/M$; $C(\mathbb{Q})/M$; or $C(\mathbb{R})/M$ is hyper-real (equivalently, non-metrizable) should be valid for *some* maximal ideal M in $C(\mathbb{N})$; $C(\mathbb{Q})$; or $C(\mathbb{R})$, respectively).

Finally, let us consider order-preserving homomorphisms between ordered fields.

In what follows, $(K, \leq)$ or $(K', \leq')$ means an ordered field, unless otherwise stated.

In $(K, \leq)$, the set $(a, b) = \{x \in K \mid a < x < b\}$ $(a < b)$ is called an *open interval*. Let $f : (K, \leq) \to (K', \leq')$ be a map. Then f is *continuous* if for each open interval $(\alpha, \beta) \subset K'$, $G = f^{-1}((\alpha, \beta))$ is *open* in K; that is, for $x \in G$, there exists an open interval $(a, b) \subset K$ such that $x \in (a, b) \subset G$. Also, f satisfies *Intermediate-Value Theorem* if for each (a, b) in K, if $\alpha \in K'$ is between $f(a)$ and $f(b)$, then there exists $c \in (a, b)$ with $f(c) = \alpha$.

Let us recall that an ordered field $(K, \leq)$ satisfies the axiom of continuity (equivalently, K is isomorphic to $\mathbb{R}$) iff all continuous maps from $(K, \leq)$ to $(K', \leq')$ satisfy Intermediate-Value Theorem (see [4], etc.).

For isomorphisms between ordered fields, the following holds (this is suggested by C. Liu).

Proposition 1. Let $f : (K, \leq) \to (K', \leq')$ be an isomorphism. Then f is order-preserving iff it satisfies Intermediate-Value Theorem.

Proof. The "only if" part is obvious. For the "if" part holds, it suffices to see that for $0 \leq a$, $0 \leq' f(a)$ (since f is a homomorphism). Suppose $f(a) <' 0$. Let $a > 1$ (without loss of generality). Since $f(1) = 1' > 0$, by Intermediate-Value Theorem, there exists $b \in K$ with $a > b > 1$ such that $f(b) = 0$. But $f(0) = 0$ and f is injective, then $b = 0$, a contradiction.

The following result is shown in [6; Theorem 2.3].

If $(K, \leq)$ is an Archimedean ordered field, then all continuous homomorphisms of $(K, \leq)$ to any $(K', \leq')$ are order-preserving.

Let us consider a question whether the converse of this result holds. We will give an example which shows that this question is negative even if we omit the continuity of the homomorphisms in the result. Then, we will give a characterization for an ordered field $(K, \leq)$ to satisfy the following condition:

(C): All homomorphisms of $(K, \leq)$ to any $(K', \leq')$ are order-preserving.

For an ordered ring $(R, \leq)$, we shall consider the following conditions (A) and (B), and define a non-negative cone $S(\leq) = \{x \in R \mid x \geq 0\}$ of R.

(A): For each $a \geq 0$ in R, there exists $b \in R$ with $a = b^2$.

(B): R is an ordered ring by only one total order $\leq$.

In (B), we can replace "total order $\leq$" by "non-negative cone S ($\leq$)" in view of Remark 1 (2).

We note that (A) implies (B) (indeed, for any non-negative cone S, $S(\leq) \subset S$ by (A), thus $S(\leq) = S$).

Remark 4. (1) $\mathbb{R}$ satisfies (A), and so does the Archimedean ordered field of all algebraic real numbers over $\mathbb{Q}$. While, $\mathbb{Q}$ (or $\mathbb{Z}$) satisfies (B), but it does not satisfy (A).

(2) Every Archimedean ordered field K need not satisfy (B), as is well-known. Indeed, let $K = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\} \subset \mathbb{R}$. Let $P = \{a + b\sqrt{2} \in K : a - b\sqrt{2} > 0\}$, where $>$ is the usual order on K from $\mathbb{R}$. Then K is an Archimedean ordered field by the usual order and a different order defined by P. (See [6; Example 3.2] as an ordered field $K' \subset \mathbb{R}$ by the usual order, and an order $\leq'$ such that $(K', \leq')$ is non-Archimedean).

Lemma 3. Any ordered ring $C(X)/I$ satisfies (A).

Proof. To see $C(X)/I$ satisfies (A), let $[f] \geq 0$ for $f \in C(X)$. Thus, we can assume $f \geq \mathbf{0}$, and define $g = \sqrt{f}$. Then $g \in C(X)$, and $f = g^2$. Thus $[f] = [g^2] = [g]^2$. Hence, $C(X)/I$ satisfies (A).

Example 5. An ordered field $(K, \leq)$ which satisfies (C), but is not Archimedean.

Proof. Let $K = (C(X)/M, \leq)$ be an ordered field which is hyper-real (hence, not Archimedean) in Remark 3. Then K satisfies (A) by Lemma 3. Thus, obviously (C) holds.

Theorem 2. For an ordered ring $(R, \leq)$, the following are equivalent.

(1) R satisfies (B).

(2) All monomorphisms of $(R, \leq)$ to any ordered ring $(R', \leq')$ are order-preserving.

Proof. For (1) $\Rightarrow$ (2), let $f : (R, \leq) \to (R', \leq')$ be a monomorphism. Let $S' = \{x \in R \mid 0 \leq' f(x)\}$. Since f is a monomorphism, S' gives a non-negative cone of R. Then, $S(\leq) = S'$ by (B). This implies that f is order-preserving. For (2) $\Rightarrow$ (1), suppose that R has a different non-negative cone S' from $S(\leq)$. Define a total order $\leq'$ on R by $a \leq' b$ if $b - a \in S'$. Then the identity map of $(R, \leq)$ to $(R, \leq')$ is not order-preserving. This is a contradiction to (2).

Corollary 2. For an ordered field $(K, \leq)$, the conditions (B) and (C) are equivalent.

Remark 5. The following result (*) also holds by Theorem 2.

(*) For an ordered integral domain $(R, \leq)$, Theorem 2 holds, replacing "ordered ring" by "ordered integral domain" in (B) of (1), and in (2).

But, the results (*) and Corollary 2 are equivalent. Indeed, this is shown by considering that for the fraction field K of an ordered integral R, there exists uniquely a positive cone P_K of K such that $P = R \cap P_K$ for a positive cone P of R.

References

[1] L. Gillman and M. Jerison, Rings of continuous functions, Van Nostrand Reinhold company, 1960.

[2] T. Y. Lam, A first course in noncommutative rings, 2nd ed., Graduate Texts in Mathematics, 131, Springer, 2001.

[3] C. Liu and Y. Tanaka, Metrizability of ordered additive groups, Tsukuba J. Math., 35(2011), 169-183.

[4] Y. Tanaka, Ordered fields and the axiom of continuity, Bull. Tokyo Gakugei Univ., Sect. 4, 46(1994), 1-6. (Japanese)

[5] Y. Tanaka, Ordered fields and the axiom of continuity. II, Bull. Tokyo Gakugei Univ., Nat. Sci., 63(2011), 1-11.

[6] Y. Tanaka, Topology on ordered fields, Comment Math. Univ. Carolin, 53(2012), 139-147.

順　序　環　と　順　序　保　存

北　村　　　好・田　中　祥　雄

数学分野

要　　旨

周知のごとく，可換環Ｒが全順序をもち，この順序関係が加法と乗法の演算に関して順序保存であるときＲは順序環と呼ばれている。とくに，Ｒが体のとき，Ｒを順序体という。

順序環や順序体を考察し，いくつかの関連事項や例を挙げる。また，順序環の剰余環が順序環や順序整域となるための特徴付けを与える。さらに，順序環Ｒが条件「Ｒから任意の順序環への全ての準同型写像が順序保存となる」を満たすための特徴付けを与える。

キーワード: 順序環，順序体，剰余環，順序保存