Bulletin of Tokyo Gakugei University, Division of Natural Sciences, **64**: 1 – 4, 2012

A note on subfields of the real number field

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(Received for Publication; May 25, 2012)

KITAMURA, Y.: A note on subfields of the real number field. Bull. Tokyo Gakugei Univ. Div. Nat. Sci., 64: 1-4 (2012)

ISSN 1880-4330

Abstract

As is well-known, the rational number field is countable but the real number field $\mathbb{R}$ is uncountable. The purpose of this note is to give a positive answer to a question in Y. Tanaka [2]: Is there an uncountable and proper subfield in $\mathbb{R}$?

Key words and phrases: real number field, subfield, algebraically independent.

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Introduction

The notations $\mathbb{R}$ and $\mathbb{Q}$ denote the real number field and the rational number field, respectively.

As is well-known, $\mathbb{Q}$ is countable but $\mathbb{R}$ is uncountable. Y. Tanaka posed the following question in [2]: Is there an uncountable and proper subfield of $\mathbb{R}$? In this note, we shall show the existence of such a subfield, which gives an affirmative answer to the question.

Results

The following assertion gives the affirmative answer in Introduction.

Assertion. *There exists an uncountable and proper subfield of* $\mathbb{R}$.

This assertion might be known, but we will show it for the reader's conveniences. First, let us recall the following.

Lemma 1. *Let D be a countable subfield of* $\mathbb{R}$. *Then there exists a real number which is not algebraic over D.*

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Proof. Let A be the set of all algebraic real numbers over D. As is well-known, A is countable (see [1], for example). Therefore, there exists a real number which is not algebraic over D. □

Definition ([1]). A finite set $\{c_1, ..., c_n\}$ of real numbers is said to be *algebraically independent* over $\mathbb{Q}$, provided that a finite sum

$$\sum a_{(\alpha_1, ..., \alpha_n)} c_1^{\alpha_1} \cdots c_n^{\alpha_n}$$

with coefficients $a_{(\alpha_1, ..., \alpha_n)}$ in $\mathbb{Q}$ and non-negative integers $\alpha_1, ..., \alpha_n$ can be zero iff all coefficients $a_{(\alpha_1, ..., \alpha_n)}$ are zero. A subset B of $\mathbb{R}$ is said to be algebraically independent over $\mathbb{Q}$, provided that an arbitrary finite subset of B is algebraically independent over $\mathbb{Q}$.

The following is a key lemma to **Assertion**.

Lemma 2. *There exists an uncountable subset B of $\mathbb{R}$ which is algebraically independent over $\mathbb{Q}$.*

Proof. Let $\mathcal{F}$ be the collection of all subsets of $\mathbb{R}$ which are algebraically independent over $\mathbb{Q}$. Obviously, $\mathcal{F}$ is not empty. Let us define a partial order on $\mathcal{F}$ by inclusion. By Zorn's Lemma, there exists a maximal member in $\mathcal{F}$. Let B be such a member in $\mathcal{F}$. If B were countable, by Lemma 1, there is some a in $\mathbb{R}$ such that a is not algebraic over B, so that $B \cup \{a\}$ is algebraically independent over $\mathbb{Q}$. This contradicts to the maximality of B in $\mathcal{F}$. Hence B is uncountable. □

Let $\{c_1, ..., c_n\}$ be a finite set of real numbers which is algebraically independent over $\mathbb{Q}$. We may assume $c_1 < \cdots < c_n$. For a non-zero polynomial

$$h = \sum a_{(\alpha_1, ..., \alpha_n)} c_1^{\alpha_1} \cdots c_n^{\alpha_n}$$

with coefficients $a_{(\alpha_1, ..., \alpha_n)}$ in $\mathbb{Q}$ and non-negative integers $\alpha_1, ..., \alpha_n$, the exponent of h is defined as follows: Let S be the collection of all exponents $(\alpha_1, ..., \alpha_n)$ of non-zero monomials $a_{(\alpha_1, ..., \alpha_n)} c_1^{\alpha_1} \cdots c_n^{\alpha_n}$ of h. Then S has an order defined by the usual lexicographic order, and the largest element of S is called the exponent of h. Let us define the leading coefficient of h as the coefficient of the monomial of h that gives the exponent of h.

Proposition 3. *Let B be an uncountable subset of $\mathbb{R}$ which is algebraically independent over $\mathbb{Q}$. Then the subfield $\mathbb{Q}(B)$ of $\mathbb{R}$ generated by B never contains $\sqrt{2}$. Therefore, $\mathbb{Q}(B)$ is an uncountable and proper subfield of $\mathbb{R}$.*

Proof. We note that such a set B exists by Lemma 2. Clearly, $\mathbb{Q}(B)$ is uncountable. Suppose the contrary that $\mathbb{Q}(B)$ contains $\sqrt{2}$. Then, we may assume that there exist polynomials $f(x_1, ..., x_n)$, $g(x_1, ..., x_n)$ with coefficients in $\mathbb{Q}$ and a finite subset $\{c_1, ..., c_n\}$ of B such that

$$\sqrt{2} = \frac{f(c_1, ..., c_n)}{g(c_1, ..., c_n)},$$

where $g(c_1, ..., c_n) \neq 0$. Hence, we have

$$2g(c_1, ..., c_n)^2 = f(c_1, ..., c_n)^2 .$$

Comparing the leading coefficients of right and left hands, we have $2b^2 = a^2$ for suitable numbers a, b in $\mathbb{Q}$, because $\{c_1, ..., c_n\}$ is algebraically independent over $\mathbb{Q}$. This is a contradiction. Therefore $\mathbb{Q}(B)$ never contains $\sqrt{2}$, completing the proof. □

Proof of **Assertion**. **Assertion** is now obvious from Proposition 3. □

Acknowledgement: The author wishes to thank Professor Y. Tanaka for his helpful comments.

References

[1] G. Birkhoff and S. Maclane, A survey of modern algebra, 3rd ed., Macmillan, 1965.

[2] Y. Tanaka, Report (2011).

実数体の部分体に関するノート

北　村　　　好

数学分野

要　旨

よく知られているように有理数体は可算であり，実数体は非可算である。本論文において，「実数体には非可算な真部分体が存在する」を示す。これは、「実数体には非可算な真部分体が存在するか？」（Y. Tanaka, Report (2011)) に対する肯定解を与えている。

キーワード: 実数体，部分体，代数的独立