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A note on subfields of the real number field

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Abstract

As is well-known, the rational number field is countable but the real number field \mathbb{R} is uncountable. The purpose of this note is to give a positive answer to a question in Y. Tanaka [2]: Is there an uncountable and proper subfield in \mathbb{R} ?

Key words and phrases: real number field, subfield, algebraically independent.

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Introduction

The notations \mathbb{R} and \mathbb{Q} denote the real number field and the rational number field, respectively.

As is well-known, \mathbb{Q} is countable but \mathbb{R} is uncountable. Y. Tanaka posed the following question in [2]: Is there an uncountable and proper subfield of \mathbb{R} ? In this note, we shall show the existence of such a subfield, which gives an affirmative answer to the question.

Results

The following assertion gives the affirmative answer in Introduction.

Assertion. *There exists an uncountable and proper subfield of \mathbb{R} .*

This assertion might be known, but we will show it for the reader's conveniences. First, let us recall the following.

Lemma 1. *Let D be a countable subfield of \mathbb{R} . Then there exists a real number which is not algebraic over D .*

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Proof. Let A be the set of all algebraic real numbers over D . As is well-known, A is countable (see [1], for example). Therefore, there exists a real number which is not algebraic over D . \square

Definition (I1). A finite set $\{c_1, \dots, c_n\}$ of real numbers is said to be *algebraically independent* over \mathbb{Q} , provided that a finite sum

$$\sum a_{(\alpha_1, \dots, \alpha_n)} c_1^{\alpha_1} \dots c_n^{\alpha_n}$$

with coefficients $a_{(\alpha_1, \dots, \alpha_n)}$ in \mathbb{Q} and non-negative integers $\alpha_1, \dots, \alpha_n$ can be zero iff all coefficients $a_{(\alpha_1, \dots, \alpha_n)}$ are zero. A subset B of \mathbb{R} is said to be algebraically independent over \mathbb{Q} , provided that an arbitrary finite subset of B is algebraically independent over \mathbb{Q} .

The following is a key lemma to **Assertion**.

Lemma 2. *There exists an uncountable subset B of \mathbb{R} which is algebraically independent over \mathbb{Q} .*

Proof. Let \mathcal{F} be the collection of all subsets of \mathbb{R} which are algebraically independent over \mathbb{Q} . Obviously, \mathcal{F} is not empty. Let us define a partial order on \mathcal{F} by inclusion. By Zorn's Lemma, there exists a maximal member in \mathcal{F} . Let B be such a member in \mathcal{F} . If B were countable, by Lemma 1, there is some a in \mathbb{R} such that a is not algebraic over B , so that $B \cup \{a\}$ is algebraically independent over \mathbb{Q} . This contradicts to the maximality of B in \mathcal{F} . Hence B is uncountable. \square

Let $\{c_1, \dots, c_n\}$ be a finite set of real numbers which is algebraically independent over \mathbb{Q} . We may assume $c_1 < \dots < c_n$. For a non-zero polynomial

$$h = \sum a_{(\alpha_1, \dots, \alpha_n)} c_1^{\alpha_1} \dots c_n^{\alpha_n}$$

with coefficients $a_{(\alpha_1, \dots, \alpha_n)}$ in \mathbb{Q} and non-negative integers $\alpha_1, \dots, \alpha_n$, the exponent of h is defined as follows: Let S be the collection of all exponents $(\alpha_1, \dots, \alpha_n)$ of non-zero monomials $a_{(\alpha_1, \dots, \alpha_n)} c_1^{\alpha_1} \dots c_n^{\alpha_n}$ of h . Then S has an order defined by the usual lexicographic order, and the largest element of S is called the exponent of h . Let us define the leading coefficient of h as the coefficient of the monomial of h that gives the exponent of h .

Proposition 3. *Let B be an uncountable subset of \mathbb{R} which is algebraically independent over \mathbb{Q} . Then the subfield $\mathbb{Q}(B)$ of \mathbb{R} generated by B never contains $\sqrt{2}$. Therefore, $\mathbb{Q}(B)$ is an uncountable and proper subfield of \mathbb{R} .*

Proof. We note that such a set B exists by Lemma 2. Clearly, $\mathbb{Q}(B)$ is uncountable. Suppose the contrary that $\mathbb{Q}(B)$ contains $\sqrt{2}$. Then, we may assume that there exist polynomials $f(x_1, \dots, x_n)$, $g(x_1, \dots, x_n)$ with coefficients in \mathbb{Q} and a finite subset $\{c_1, \dots, c_n\}$ of B such that

$$\sqrt{2} = \frac{f(c_1, \dots, c_n)}{g(c_1, \dots, c_n)},$$

where $g(c_1, \dots, c_n) \neq 0$. Hence, we have

$$2g(c_1, \dots, c_n)^2 = f(c_1, \dots, c_n)^2.$$

Comparing the leading coefficients of right and left hands, we have $2b^2 = a^2$ for suitable numbers a, b in \mathbb{Q} , because $\{c_1, \dots, c_n\}$ is algebraically independent over \mathbb{Q} . This is a contradiction. Therefore $\mathbb{Q}(B)$ never contains $\sqrt{2}$, completing the proof. \square

Proof of Assertion. **Assertion** is now obvious from Proposition 3. \square

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References

- [1] G. Birkhoff and S. MacLane, A survey of modern algebra, 3rd ed., Macmillan, 1965.
- [2] Y. Tanaka, Report (2011).

実数体の部分体に関するノート

北 村 好

数学分野

要 旨

よく知られているように有理数体は可算であり、実数体は非可算である。本論文において、「実数体には非可算な真部分体が存在する」を示す。これは、「実数体には非可算な真部分体が存在するか？」(Y. Tanaka, Report (2011))に対する肯定解を与えている。

キーワード: 実数体, 部分体, 代数的独立