Bulletin of Tokyo Gakugei University, Division of Natural Sciences, 68: pp. 13～23, 2016

Product Extension Rings and Partially Ordered Rings

Yoshimi KITAMURA* and Yoshio TANAKA*

Department of Mathematics

(Received for Publication; May 25, 2016)

KITAMURA, Y. and TANAKA, Y.: Product extension rings and partially ordered rings. Bull. Tokyo Gakugei Univ. Div. Nat. Sci., 68: 13-23 (2016) ISSN 1880-4330

Abstract

As an extension of a commutative ring R, we introduce the notion of product extension rings on the product $R \times R$ in [5] (or [4]), and we consider algebraic structures of these rings in [5]. Concerning order structures of rings, as a generalization of positive cones, we introduce the notion of non-negative semi-cones ([3]), and we use the abbreviation "semi-cone" for "non-negative semi-cone" ([4]). These semi-cones determine the partial orders on rings. In [4], we consider order structures of basic product rings by canonical semi-cones. In this paper, we consider order structures of the product extension rings by certain semi-cones. Namely, we give characterizations for the extension rings to be partially ordered rings with these semi-cones. Also, we consider convex ideals in these partially ordered rings. In particular, we give characterizations of the convexity of ideals in the partially ordered rings with a lexicographic order. As its application, in the trivial extension of the ring of integers by itself, we determine precisely convex ideals with respect to a lexicographic order and the number of the convex ideals.

Keywords: product extension ring, direct product ring, partially ordered ring, semi-cone, convex ideal, lexicographic order

Department of Mathematics, Tokyo Gakugei University, 4-1-1 Nukuikita-machi, Koganei-shi, Tokyo 184-8501, Japan

1. Introduction

The symbol R means a non-zero commutative ring with the identity element denoted by 1. The symbol $\mathbb{N}$; $\mathbb{Z}$; $\mathbb{R}$ means respectively the set of natural numbers; the ring of integers; the field of real numbers.

In Section 2, we consider certain semi-cones on the product extension rings of partially ordered rings. The concept of semi-cones plays an important role in the study of partially ordered rings ([2, 3, 4]). As is well-known, for a ring R, there exists a bijection between the set of partial orders of R which make R a partially ordered ring and the set of semi-cones of R. We deal with several semi-cones of the product extension rings of partially ordered rings. In particular, we study the lexicographic order, and a certain order which gives some useful examples related to convex ideals or semi-cones (see [1, 4]). We obtain characterizations for the product extension rings to be partially ordered rings with respect to these orders, and we show that the

* Tokyo Gakugei University (4-1-1 Nukuikita-machi, Koganei-shi, Tokyo, 184-8501, Japan)

product extension ring of any ordered field is not an ordered field by the lexicographic order.

As is well-known, for a partially ordered ring R, every convex ideal of R is characterized as a kernel of an order-preserving ring homomorphism from R to another partially ordered ring. In Section 3, we consider the convexity of ideals in the product extension rings with respect to several semi-cones. In particular, we give characterizations of the convexity of ideals in the extension ring with a lexicographic order. Further, for a principal ideal domain R, we determine the convex ideals with respect to a lexicographic order in the trivial extension of R by itself. As its application, for R being the ring of integers, we determine precisely convex ideals with respect to the order in the trivial extension of R, and the number of the convex ideals.

2. Semi-cones

In this paper, the symbol $A \times B$ means the product of sets A and B. Let A, B be subsets of R. Define $-A = \{-x \mid x \in A\}$, $A + B = \{x + y \mid x \in A, y \in B\}$, $AB = \{xy \mid x \in A, y \in B\}$, $aB = \{a\}B$ for $a \in R$, and define $A \backslash \{0\} = \{x \mid x \in A, x \neq 0\}$.

The single set $\{0\}$ is often denoted by 0.

Let us recall main terminologies and notations used in this paper.

Let $P = R \times R$. For each $(a, b) \in P$, the product P is a commutative ring under the following addition and multiplication:

$$\begin{aligned}(x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2)\\ (x_1, y_1) * (x_2, y_2) &= (x_1x_2 + ay_1y_2, x_1y_2 + y_1x_2 + by_1y_2).\end{aligned}$$

We note that $e = (1, 0)$ is the identity element of the ring P, and for $u = (0, 1)$, $u * u = (a, b)$, and $(x, y) = (x, 0) * e + (y, 0) * u$ for any $(x, y) \in P$.

The ring P is a commutative R-algebra which contains a subring isomorphic to R, and it gives useful ring-theoretic constructions or examples. We shall call such an extension ring the *product extension ring* of R, and denote it by $(R \ltimes R; a, b)$ ([5]). For example, $(\mathbb{R} \ltimes \mathbb{R}; -1, 0)$ is a field isomorphic to the field of complex numbers. Especially, a basic extension ring $(R \ltimes R; 0, 0)$ is said to be the *trivial extension* of R by itself. Let us denote $(R \ltimes R; 0, 0)$ by $R \ltimes R$. As is well-known, this ring gives useful examples related to ring structures and order structures, or extensions. Algebraic structures of the rings $(R \ltimes R; a, b)$ and their ideals are observed in [5].

As is well-known, the product $P = R \times R$ is also a commutative ring under the component-wise addition and multiplication, that is, $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$, $(x_1, y_1) \bullet (x_2, y_2) = (x_1x_2, y_1y_2)$. This canonical ring P with the identity element $(1, 1)$ is called the *direct product ring* of R, and let us denote the ring by $R \otimes R$. The ring $R \otimes R$ contains a subring isomorphic to R, but it is never an integral domain.

For a partial order $\leq$ on R, $(R, \leq)$ is a *partially ordered ring* ([1]) if R satisfies the following conditions:

(i) $a \leq b$ implies $a + x \leq b + x$ for all x, and

(ii) $a \leq b$ and $0 \leq x$ implies $ax \leq bx$.

For a partially ordered ring $(R, \leq)$, if the partial order $\leq$ is a total order, then such an $(R, \leq)$ is called a *totally ordered ring* (abbreviated to *ordered ring*), in particular, for R being a field (resp. integral domain), such an ordered ring $(R, \leq)$ is called an *ordered field* (resp. *ordered integral domain*).

For a subset S of a ring R, we shall call S a *non-negative semi-cone* ([3]) of R if S satisfies (i), (ii), and (iii) below, and call S a *non-negative cone* ([2]) if S satisfies (i), (ii), (iii), and (iv). We recall that a subset S of R satisfying (ii) and (iii) is a *positive-cone* if $R \backslash \{0\} = S \cup (-S)$ ([7] (or [2])).

(i) $S \cap (-S) = 0$,

(ii) $S + S \subset S$,

(iii) $SS \subset S$, and

(iv) $R = S \cup (-S)$.

We will use the brief terminologies "*semi-cone*" (resp. "*cone*") as an abbreviation of "non-negative semi-cone" (resp. "non-negative cone"), as in [4].

Evidently, the intersection of any (non-empty) family of semi-cones of R is a semi-cone.

We note that for a semi-cone S of a ring R, we induce a partially ordered ring $(R, \leq_S)$, defining $x \leq_S y$ by $y - x \in S$ (thus, for a partially ordered ring $(R, \leq)$, putting $S = \{x \in R \mid 0 \leq x\}$, S is a semi-cone of R, and $\leq \, = \, \leq_S$).

In view of the above, a ring R is a partially ordered ring; ordered ring; ordered integral domain iff it has a semi-cone; cone; positive cone, respectively.

In this paper, the symbol S means a *non-zero* semi-cone of R, and the symbol S_0 means the (non-empty) set $S\backslash\{0\}$, unless otherwise stated.

For S of R, let us define the following subsets of the product set $R \times R$. (Let us call the set D_0 (resp. L) below the *diagonal set* (resp. *lexicographic set*) (by S)).

$D_0 = \{(s, s) \mid s \in S\}$ $(\subset S \times S)$.

$D_1 = \{(s + s', s) \mid s, s' \in S\}$.

$D_2 = \{(s, s + s') \mid s, s' \in S\}$.

$L_0 = S \times S$

$L = L_0 \cup (S_0 \times R)$ $(= (S_0 \times R) \cup (0 \times S))$.

$L' = L_0 \cup (R \times S_0)$ $(= (R \times S_0) \cup (S \times 0))$.

Obviously, $D_0 = D_1 \cap D_2$, and $D_1, D_2 \subset L_0 = L \cap L'$.

In [4], the sets L_0, L, L', and other typical subsets of $R \times R$ are considered in the rings $R \otimes R$ and $R \ltimes R$. The sets D_0, D_1, D_2, and L_0 are obviously semi-cones in $R \otimes R$, but neither L nor L' is a semi-cone in $R \otimes R$ ([4]). We will consider these subsets in the product extension ring $(R \ltimes R; a, b)$. Clearly, $D_1 = D_0 + (S \times 0)$, $D_2 = D_0 + (0 \times S)$ and $L_0 = D_1 + D_2$ in $R \otimes R$ or $(R \ltimes R; a, b)$.

For S of R, let us denote the partial order $\leq_S$ by $\leq$. For the set L (resp. D_1) being a semi-cone in the ring $(R \ltimes R; a, b)$, let us denote the partial order $\leq_L$ (resp. $\leq_{D_1}$) by $\preceq$ (resp. $\leq_*$). We shall call the order $\preceq$ *the lexicographic order*. The order $\leq_*$ is given on the ring $\mathbb{R} \otimes \mathbb{R}$ in [1, 5B].

Theorem 2.1. *For S of R, the following hold in $(R \ltimes R; a, b)$.*

(1) D_0 is a semi-cone iff (1.1) $(a + 1)SS \subset S$, and (1.2) $(a - b - 1)SS = 0$.

(2) D_1 is a semi-cone iff (2.1) $(b + 2)SS \subset S$, and (2.2) $(a - b - 1)SS \subset S$.

(3) D_2 is a semi-cone iff (3.1) $aSS \subset S$, and (3.2) $(b - a)SS \subset S$.

(4) L_0 is a semi-cone iff (4.1) $aSS \subset S$, and (4.2) $bSS \subset S$.

(5) L is a semi-cone iff (5.1) $aS = bSS = 0$, (5.2) $S_0S_0 + aR \subset S_0$, and (5.3) $(S_0 + bR)S \subset S$.

(6) L' is never a semi-cone (for any a, b).

Proof. (1) and (4) are routinely shown.

For (2), recall that $D_1 = \{(s + t, s) \mid s, t \in S\}$ and that $((R \ltimes R; a, b), \leq_*)$ is a partially ordered ring iff $D_1 * D_1 \subset D_1$. Let

$$p = (s + s_1, s),\ q = (t + t_1, t)\ (s, s_1, t, t_1 \in S).$$

Put $(c, d) = p * q \in (R \ltimes R; a, b)$. Then,

$$c = (a + 1)st + s_1t + st_1 + s_1t_1, \text{ and } d = (b + 2)st + s_1t + st_1.$$

Thus, $c - d = (a - b - 1)\, st + s_1t_1$. Hence, $p * q \in D_1$ iff $c \geq d \geq 0$. While, $c \geq d \geq 0$ iff $(b + 2)\, st \geq 0$ and $(a - b - 1)st \geq 0$ (for all $s, t \in S$) (actually, the if part is obvious. For the only if part, put $s_1 = t_1 = 0$), proving (2).

For (3), to see the only if part, let $p = (s, s + s_1)$, $q = (t, t + t_1)$ for $s, s_1, t, t_1 \in S$, and let $p * q = (c, d)$. Then

$$c = (a + 1)st + a(st_1 + s_1t) + as_1t_1, \text{ and } d = (b + 2)\, st + (b + 1)(st_1 + s_1t) + bs_1t_1.$$

Thus $d - c = (b + 1 - a)(st + st_1 + s_1t) + (b - a)\, s_1t_1$. While, $d \geq c \geq 0$; that is,

$$(a + 1)\, st + (st_1 + s_1t) + as_1t_1 \geq 0, \text{ and } (b + 1 - a)(st + st_1 + s_1t) + (b - a)s_1t_1 \geq 0.$$

Putting $s = t = 0$, $as_1t_1 \geq 0$ and $(b - a)s_1t_1 \geq 0$ (for all $s_1, t_1 \in S$). Hence, the only if part holds. Conversely, the if part is obvious.

For (5), we will show the only if part. First, let us show $S_0S_0 \subset S_0$. Suppose the contrary that there exist $x, x' \in S_0$ such that $xx' \notin S_0$. Then $xx' = 0$, and $(x, -1), (x', 0) \in L$, but $(x, -1) * (x', 0) = (0, -x') \notin L$. Hence L is not closed under multiplication in $(R \ltimes R;\, a, b)$, a contradiction. Thus $S_0S_0 \subset S_0$. Now, to see $aS = 0$ in (5.1), let $x > 0$ and $s \geq 0$. Since $(x, 1) * (0, s) = (as, xs + bs) \in L$, we have $as \geq 0$. Similarly, from $(x, -1) * (0, s) = (-as, xs - bs) \in L$, we have $-as \geq 0$. Thus we have $as = 0$, which yields $aS = 0$. To see $bSS = 0$ in (5.1), let $s \geq 0$, $s' \geq 0$. Then $(0, s), (0, s') \in L$. Noting $aS = 0$, we have $(0, s) * (0, s') = (0, bss') \in L$, which yields $bss' \geq 0$. Assume now $bss' > 0$ for some $s, s' \in S$. Let $s'' > 0$. Then $(bss')\, s'' > 0$ by $S_0S_0 \subset S_0$. Since $(bss', -2ss'), (0, s'') \in L$ and $aS = 0$, we have $(bss', -2ss') * (0, s'') = (0, -bss's'') \in L$, which yields $-bss's'' \geq 0$, a contradiction. Therefore $bSS = 0$. Hence (5.1) holds. To see (5.2) $S_0S_0 + aR \subset S_0$, let $x > 0$, $z > 0$ and $y, w \in R$. Then $(x, y), (z, w) \in L$. Since $L * L \subset L$, we have $(x, y) * (z, w) = (xz + ayw, xw + yz + byw) \in L$. Thus $xz + ayw \geq 0$. If $xz + ayw = 0$ for some $x > 0$, $z > 0$, y, w, then $x\,(xz + ayw) = x^2z = 0$ by $aS = 0$, but $x^2z > 0$ by $S_0S_0 \subset S_0$, a contradiction. Hence $xz + ayw > 0$ for any $x, z \in S_0$ and $y, w \in R$, that is, $S_0S_0 + aR \subset S_0$. To see (5.3) $(S_0 + bR)\, S \subset S$, let $x > 0$, $w \geq 0$ and $y \in R$. Then $(x, y), (0, w) \in L$. Since $L * L \subset L$, we have $(x, y) * (0, w) = (ayw, xw + byw) = (0, xw + byw) \in L$, which yields $xw + byw \geq 0$. Thus $(S_0 + bR)\, S \subset S$ holds. Next, we will show the if part. Obviously, $L + L \subset L$ and $L \cap (-L) = 0$ hold. Let $L_1 = S_0 \times R$, $L_2 = 0 \times S$. Then $L_1 * L_1 \subset L_1$ by (5.2), and $L_1 * L_2 \subset L_2$ by (5.1) and (5.3). Further, $L_2 * L_2 = 0$ holds by (5.1). Thus $L * L \subset L$. Hence L is a semi-cone of $(R \ltimes R;\, a, b)$.

For (6), suppose L' is a semi-cone of $(R \ltimes R;\, a, b)$. Let $s \in S_0$. Then $(-2bs, s) * (0, s) = (as^2, -bs^2) \in L'$, thus $-bs^2 \in S$. Also, $(0, s)^2 = (as^2, bs^2) \in L'$, thus $bs^2 \in S$. Hence, $bs^2 = 0$. While, $(-1, s), (0, s) \in L'$, but $(-1, s) * (0, s) = (as^2, -s) \notin L'$, a contradiction. □

Remark 2.2. *Let L be a semi-cone of* $(R \ltimes R;\, a, b)$. *Then (i)* $-xz < a < xz$ *for any* $x, z \in S_0$, *and (ii)* $a^2 = 0$ *in Theorem 2.1(5) (indeed, (i) holds by (5.2), then* $xz + a \in S$ *for* $x, z \in S_0$. *Hence (5.1) implies* $a(xz + a) = a^2 = 0$). *However, the authors do not know whether* $a = 0$ *holds in (ii).*

The conditions (5.1), (5.2) and (5.3) in Theorem 2.1 are independent by the following example. Let $\mathbb{Z}^* = \mathbb{N} \cup 0$.

Example 2.3. (1) Let $\mathbb{Z}[X]$ be the polynomial ring over $\mathbb{Z}$, and $I = (X^3 - X^2)$ the ideal of $\mathbb{Z}[X]$ generated by $X^3 - X^2$. Let $R = \mathbb{Z}[X]/I$ be the residue class ring, and $x = [X] \in R$. Let $S = x\mathbb{Z}^* + x^2\mathbb{Z}^*$. Let $a = 0$, $b = x - 1 \in R$. Then $x^2 = x^3 = x^4$, $SS = x^2\mathbb{Z}^*$, and S is a semi-cone of R with $S_0S_0 \subset S_0$. Obviously, $aS = 0$, $bSS = 0$, and $S_0S_0 + aR \subset S_0$, thus (5.1) and (5.2) hold. But, (5.3) does not hold, since $(x + b)\, x = 2x^2 - x \notin S$.

(2) Let $R = \mathbb{Z} \ltimes \mathbb{Z}$ and $S = (\mathbb{N} \times \mathbb{Z}) \cup \{(0, 0)\}$. Let $a = (0, 0)$, $b = (0, 1) \in R$. Obviously, S is a semi-cone of R, and $S_0S_0 = S_0$.

Since $S_0S_0 + aR = S_0$ and $S_0 + bR = S_0$, (5.2) and (5.3) hold. But $bSS \neq 0$ because $(1, 0) \in S$. Hence, (5.1) does not hold.

(3) Let $R = \mathbb{Z} \otimes \mathbb{Z}$ be the direct product ring, and $m \in \mathbb{N}$. Let $S = m\mathbb{Z}^* \times 0$. Evidently, S is a semi-cone of R with $S_0S_0 \subset S_0$. Let $a = b = (0, 1) \in R$. Obviously, $aS = bS = 0$. Thus (5.1) and (5.3) hold. But (5.2) does not hold, since $(m, 0)(m, 0) + a = (m^2, 1) \notin S$.

The following holds by the proof of Theorem 2.1(5).

Corollary 2.4. *L is a semi-cone of $(R \ltimes R; a, b)$ iff $L_1 * L_1 \subset L_1$, $L_1 * L_2 \subset L_2$ and $L_2 * L_2 = 0$ hold, where $L_1 = S_0 \times R$, $L_2 = 0 \times S$.*

We will give other corollaries of Theorem 2.1. For a semi-cone S of R ($S \neq 0$), let us define a subset $\mathrm{ann}(S) = \{x \in R \mid xS = 0\}$ of R.

Let p_1, p_2: $R \times R \to R$ be the projections defined by $p_1(x, y) = x$, and $p_2(x, y) = y$, unless otherwise stated.

Remark 2.5. *(1) R is an ordered ring (generally, $1 \in S$) or an integral domain $\Rightarrow$ $\mathrm{ann}(S) = 0 \Rightarrow SS \neq 0$. These converses need not hold.*

(2) For a semi-cone S of R, there exists a non-zero semi-cone $S' \subset S$ of R with $S'S' = 0$ iff (*) $pp = 0$ for some $p \in S_0$. It is impossible to replace (*) by (**): $pq = 0$ for some $p, q \in S_0$ with $p \neq q$.

(3) A subset $A = 0 \times S$ of $R \ltimes R$ is a semi-cone with $A * A = 0$. More generally, for a subset A of $R \ltimes R$, (b) $\Leftrightarrow$ (c) $\Rightarrow$ (a) below holds. For R being an integral domain, (a), (b), (c) are equivalent. (Note that (c) need not imply $p_2(A)$ is a semi-cone of R (by a semi-cone $A = 0 \times (-\mathbb{Z}^*)$ of $\mathbb{Z} \ltimes \mathbb{Z}$)).

(a) A is a semi-cone with $A * A = 0$; (b) A is a semi-cone with $A = 0 \times p_2(A)$; and (c) $p_1(A) = 0$, $p_2(A) \cap p_2(-A) = 0$, and $p_2(A) + p_2(A) \subset p_2(A)$.

Indeed, for (1), the implications are obvious. The first (resp. second) converse need not hold by a non-zero semi-cone $S' = S \times S$ with $1 \notin S$ (resp. $S' = S \times 0$) of $R' = R \otimes R$ with R an integral domain. For (2), the only if part is obvious. For the if part, if $pS \neq 0$, let $S' = pS$, otherwise, let $S' = \{x \in S \mid xS = 0\}$ (= $\mathrm{ann}(S) \cap S$). Then $S' \subset S$ is a non-zero semi-cone of R with $S'S' = 0$. For the latter part, for an integral domain R, $L_0 = S \times S$ is a semi-cone of $R' = R \otimes R$ satisfying (**), but $S'S' \neq 0$ for any non-zero semi-cone S' of R'. (3) is routinely shown.

Corollary 2.6. *For S of R, the following hold in $(R \ltimes R; a, b)$.*

(1) (a) D_0 is a semi-cone iff D_1 is a semi-cone with $(a - b - 1)\, SS = 0$.

(b) D_0 and D_2 are semi-cones $\Leftrightarrow$ D_1 and D_2 are semi-cones $\Leftrightarrow$ $SS = 0 \Rightarrow L_0$ is a semi-cone. Thus, if D_2 is a semi-cone with $\mathrm{ann}(S) = 0$, then neither D_0 nor D_1 is a semi-cone.

(2) Let L be a semi-cone. Then D_2 and L_0 are semi-cones, but neither D_0 nor D_1 is a semi-cone.

Proof. The symbol (i, j) means condition (i, j) in Theorem 2.1. In (1), for (a), note that (1.1) $\Leftrightarrow$ (2.1) under (1.2), actually, for $s, t \in S$, $(b + 2)\, st = bst + 2st = (ast - st) + 2st = ast + st = (a + 1)\, st$. For (b), we show that $SS = 0$ if D_0 and D_2 are semi-cones. By (1.2), $(a - b)\, SS \subset S$. But, by (3.2), $(a - b)\, SS \subset -S$. Thus, $(a - b)\, SS = 0$, then $-SS = 0$ by (1.2), so $SS = 0$. For D_1 and D_2 being semi-cones, $-SS \subset S$ by (2.2) and (3.2), so $SS = 0$. For (2), D_2 and L_0 are semi-cones, using (5.1), but neither D_0 nor D_1 is a semi-cone by (1)(b) since $S_0S_0 \neq 0$ by (5.2). □

The following holds by (2), (4) and (5) in Theorem 2.1, for example.

Corollary 2.7. *For S of R, the following hold in $(R \ltimes R; a, b)$.*

(1) If $a \geq b + 1 \geq -1$ in R, then D_1 is a semi-cone, and the converse holds if $S \ni 1$.

(2) If $a, b \in S$, *then* L_0 *is a semi-cone, and the converse holds if* $S \ni 1$.

(3) (a) If L *is a semi-cone, then* $S_0S_0 \subset S_0$, *and the converse holds if* $a = b = 0$.

(b) Assume $\mathrm{ann}(S) = 0$, *or* $a, b \in S \cup -S$. *Then* L *is a semi-cone iff* $S_0S_0 \subset S_0$ *and* $a = b = 0$. *For* R *being an integral domain or* $S \ni 1$, L *is a semi-cone iff* $a = b = 0$.

Remark 2.8. *(1) In (1) and (2) of Corollary 2.7, the converses need not hold if* $S \ni 1$ *is deleted. For (3), in (a), the converse need not hold if* $a = b = 0$ *is deleted. In (b), the only if part need not hold if the assumption is deleted.*

(2) For a partially ordered ring R, *(i)* $S_0S_0 \subset S_0$ *and (ii)* $\mathrm{ann}(S) = 0$ *in* R *are independent.*

Indeed, for (1), the first is shown by Corollary 2.6(1)(b), considering $SS = 0$ as in Remark 2.5. The second is shown in Example 2.3(3). The last is shown by this example, but take $a = (0, 0)$, $b = (0, 1)$, then $aR = bS = 0$ and $S_0S_0 \subset S_0$, thus L is a semi-cone by Theorem 2.1(5), but $b \neq 0$. For (2), the previous example also shows that (i) need not imply (ii). (ii) need not imply (i) by a cone $S' = (\mathbb{N} \times \mathbb{Z}) \cup (0 \times \mathbb{Z}^*)$ of $R' = \mathbb{Z} \ltimes \mathbb{Z}$ (in fact, (ii) holds in R' by $e = (1, 0) \in S'$, but (i) does not hold by $u = (0, 1) \in S'_0$ with $u * u = (0, 0)$ in R').

The following is a consequence of Corollary 2.7(3), noting that the partial order $\preceq$ on $(R \ltimes R; a, b)$ is total if the partial order $\leq$ on R is total.

Proposition 2.9. *Let* $(R, \leq)$ *be an ordered ring. Then* $((R \ltimes R; a, b), \preceq)$ *is an ordered ring iff* R *is an integral domain and* $a = b = 0$.

We note that for an ordered field K with $a < 0$, $(K \ltimes K; a, 0)$ is a field ([5]), but it is not an ordered field by the lexicographic order L. Indeed, we have the following as a consequence of Proposition 2.9, noting that $R \ltimes R$ is never an integral domain.

Proposition 2.10. *For any ordered integral domain* R, *any* $((R \ltimes R; a, b), \preceq)$ *is never an ordered integral domain. In particular, for any ordered field* K, *any* $((K \ltimes K; a, b), \preceq)$ *is not an ordered field.*

Let X be a topological space. Recall that $C(X)$ is the usual ring of all continuous maps from X into the usual space $\mathbb{R}$. The constant map $\mathbf{1}$ (i.e., $\mathbf{1}(x) = 1$ for all $x \in X$) is the identity element of $C(X)$. The commutative ring $C(X)$ has the usual partial order $\leq$ (i.e., $f, g \in C(X)$, $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in X$). Obviously, $\mathbf{0} < \mathbf{1}$. We have the following, for example.

Proposition 2.11. *Let* $R = (C(X), \leq)$ *be the partially ordered ring. Then the following hold.*

(1) D_1 *is a semi-cone of* $(R \ltimes R; f, g)$ *iff* $f \geq g + 1 \geq -1$. *In particular,* D_1 *is never a semi-cone of* $R \ltimes R$.

(2) L_0 *is a semi-cone of* $(R \ltimes R; f, g)$ *iff* $f \geq \mathbf{0}$ *and* $g \geq \mathbf{0}$.

(3) If X *is a completely regular space containing more than one point, then* L *is never a semi-cone of* $(R \ltimes R; f, g)$ *for any* $f, g \in R$.

Proof. The results hold by Corollary 2.7. Indeed, for (3), since X is completely regular, R contains non-zero elements f_1, g_1 such that $f_1g_1 = \mathbf{0}$ ([1, 3A]). Then $|f_1||g_1| = \mathbf{0}$, and $|f_1| > \mathbf{0}$, $|g_1| > \mathbf{0}$. Hence (3) follows from Corollary 2.7(3). □

For a field K, we give a characterization for cones of $K \ltimes K$ in [4]. For the ring $\mathbb{Z}$, we have the following determination of cones in $\mathbb{Z} \ltimes \mathbb{Z}$.

Proposition 2.12. *The cones of* $\mathbb{Z} \ltimes \mathbb{Z}$ *are precisely the sets* $L = L_1 \cup L_2$ *and* $L^* = L_1 \cup -L_2$, *where* $L_1 = \mathbb{N} \times \mathbb{Z}$, $L_2 = 0 \times \mathbb{Z}^*$.

Proof. Routinely, L and L^* are cones of $\mathbb{Z} \ltimes \mathbb{Z}$. Let A be a cone of $\mathbb{Z} \ltimes \mathbb{Z}$. Then (i) $(1, 0) \in A$, (ii) $(1, 1) \in A$, and (iii) $(1, -1) \in A$ hold. Indeed, (i) is obvious by $(1, 0) * (1, 0) = (1, 0)$. For (ii), assume $(1, 1) \notin A$, then $(-1, -1) \in A$, and hence $(0, -1) =$

$(1, 0) + (-1, -1) \in A$. But $(0, 1) = (-1, -1) * (0, -1) \in A$. Hence $(0, 1) \in A \cap -A = 0$, a contradiction. Thus $(1, 1) \in A$. For (iii), assume $(1, -1) \notin A$. Then $(-1, 1) \in A$, and hence $(0, 1) = (1, 0) + (-1, 1) \in A$. But $(0, -1) = (0, 1) * (-1, 1) \in A$. Hence $(0, -1) \in A \cap -A$, a contradiction. Thus $(1, -1) \in A$. By (i), (ii) and (iii), we have $L_1 \subset A$ (in fact, let $m, n \in \mathbb{N}$. Then $m(1, 0) = (m, 0) \in A$. Also, $(1, 1)^n = (1, n) \in A$. Hence $(m-1)(1, 0) + (1, 1)^n = (m, n) \in A$. Similarly, $(m, -n) \in A$. Thus $L_1 \subset A$). For a case $(0, 1) \in A$, $L_2 \subset A$. Hence $A = L$. For a case $(0, 1) \in -A$, $-L_2 \subset A$. Hence $A = L^*$. □

Proposition 2.13. *For a semi-cone S' of $R \ltimes R$, the following hold.*

(1) $p_1(S')$ is a semi-cone of R iff $p_1(S') \cap p_1(-S') = 0$. For R being an integral domain, $p_1(S')$ is a semi-cone.

(2) $p_2(S')$ is a semi-cone of R iff $p_2(S') \cap p_2(-S') = 0$, and $p_2(S')\, p_2(S') \subset p_2(S')$.

Proof. For (1), the first half is routinely shown. For the latter part, assume R is an integral domain. To see $p_1(S') \cap p_1(-S') = 0$, let $s \in p_1(S') \cap p_1(-S')$. Then there are some $t, t' \in R$ such that $(s, t), (-s, -t') \in S'$. Thus $(s, t) + (-s, -t') = (0, t - t') \in S'$. Hence, $(s, t) * (0, t - t') \in S'$ and $(-s, -t') * (0, t - t') \in S'$, thus $(0, s(t - t')) \in S' \cap -S'$. Hence $s = 0$ or $t = t'$, which yields $s = 0$. (2) is obvious. □

Remark 2.14. *(1) In Proposition 2.13, for a semi-cone S' of $R \ltimes R$,*

(a) $p_1(S')$ need not be a semi-cone in (1) if $p_1(S') \cap p_1(-S') = 0$ is deleted ([6]).

(b) $p_2(S')$ need not be a semi-cone in (2) if $p_2(S') \cap p_2(-S') = 0$, or $p_2(S')\, p_2(S') \subset p_2(S')$ is deleted (by a semi-cone $S' = L$, or $S' = 0 \times (-\mathbb{Z}^*)$ in $\mathbb{Z} \ltimes \mathbb{Z}$, respectively).

(2) A subset A of $R \ltimes R$ need not be a semi-cone even if $p_1(A) \times p_2(A)$ is a semi-cone, and $p_1(A)$ and $p_2(A)$ are semi-cones of R (by $A = (\mathbb{Z}^* \times 0) \cup (0 \times \mathbb{Z}^*)$ in $\mathbb{Z} \ltimes \mathbb{Z}$).

3. Convex ideals

Let I be an ideal of a partially ordered ring $(R, \leq)$ with I proper (i.e., $I \neq R$). We recall that I is *convex* if whenever $0 \leq x \leq y$, and $y \in I$, then $x \in I$. Routinely, ann(S) is a convex ideal for a (non-zero) semi-cone S. Also, the intersection of any (non-empty) family of convex ideals is convex. The convexity of I in R gives a characterization for the residue class rings R/I to be a partially ordered ring with the natural order induced by the partial order $\leq$ in R ([1, 5.2]). For convex ideals in partially ordered rings, see [1, 3, 4], etc.

For an ideal I of R, we assume I need not be proper, but let us consider the convexity of I under I being proper. Also, let us say that an ideal I is convex with respect to T ($\subset$ R) if I is convex in ($R, \leq_T$) under assuming T is a semi-cone in R.

Let us recall the following basic fact on ideals of the ring $R \otimes R$ or $(R \ltimes R; a, b)$. (1) is well-known (or routinely shown). (2) is shown in [5], which will be often used.

Fact 3.1. *(1) For an ideal I of $R \otimes R$, $I = p_1(I) \times p_2(I)$ with $p_1(I)$, $p_2(I)$ ideals of R.*

(2) (a) Let I and J be ideals of R. Then $I \times J$ is an ideal of $(R \ltimes R; a, b)$ iff $aJ \subset I \subset J$. In particular, for $a \in I$, $I \times J$ is an ideal of $(R \ltimes R; a, b)$ iff $I \subset J$. Also, $I \times I$ is an ideal (but, $I \times 0$ is never an ideal for $I \neq 0$).

(b) For an ideal I of $(R \ltimes R; a, b)$, $p_1(I) \times p_2(I)$ is an ideal of $(R \ltimes R; a, b)$ with $p_1(I)$, $p_2(I)$ ideals of R (but $I = p_1(I) \times p_2(I)$ need not hold).

In [4], we consider the convexity of ideals of $R \otimes R$ with respect to T (such as $T = S \times S$, or $T = (S_0 \times A) \cup \{(0, 0)\}$, where $A = 0, S_0, S$, or R). Also, the following routinely holds in $R \otimes R$, for example.

Proposition 3.2. *Let I, J be ideals of R with $I \neq R$. If $I \times J$ is convex with respect to a semi-cone D_1 in $R \otimes R$, then I is convex.*

The converse holds if J is convex or $I \subset J$.

Let us consider the convexity of ideals of $(R \ltimes R; a, b)$ with respect to T, where $T = D_0, D_1, D_2, L_0$, or L (as in Theorem 2.1).

Proposition 3.3. *Let I, J be ideals of R, and let $I \times J$ be an ideal of $(R \ltimes R; a, b)$ (as in Fact3.1(2)). Then the following hold.*

(1) $I \times J$ is convex with respect to D_0 iff I is convex in R.

(2) $I \times J$ is convex with respect to D_1 iff I is convex in R.

(3) $I \times J$ is convex with respect to D_2 iff I is convex in R, and J is convex in R or $J = R$.

(4) $I \times J$ is convex with respect to L_0 iff I is convex in R, and J is convex in R or $J = R$.

(5) $I \times J$ is convex with respect to L iff I is convex in R, and either J is convex and $I \cap S_0 = \emptyset$, or $J = R$.

Proof. First, note that $I \times J$ is proper iff so is I (by $I \times J \ni e = (1, 0)$ iff $I \ni 1$). Noting $I \subset J$, it is routine to see (1) ~ (4). (for the if part of (2), note $(0, 0) \leq_* (x, y) \leq_* (z, w)$ implies $0 \leq y \leq x \leq z$).

For (5), assume $I \times J$ is convex. Evidently, I is convex, and J is convex if $J \neq R$. Let $J \neq R$. To see $I \cap S_0 = \emptyset$, suppose not. Take $x \in I \cap S_0$, then $(0, 0) \preceq (x, 1) \preceq (2x, 0) \in I \times J$, so $1 \in J$, a contradiction. Thus $I \cap S_0 = \emptyset$. For the converse, to see $I \times J$ is convex, let $(0, 0) \preceq (x, y) \preceq (z, w) \in I \times J$. For $J = R$, $(x, y) \in I \times J$ by the convexity of I. For $J \neq R$, since $I \cap S_0 = \emptyset$, $x = z = 0$. But, $0 \leq y \leq w \in J$, thus $y \in J$ by the convexity of J. Hence $(x, y) = (0, y) \in I \times J$. Thus $I \times J$ is convex. □

Let us give ideals which are convex or not in a partially ordered ring $(R \ltimes R; a, b)$ with the partial order $\leq_*$ $(= \leq_{D1})$ or $\preceq$ $(= \leq_L)$.

Example 3.4. Let R be the ring, and S be the semi-cone of R in Example 2.3(3) (that is, $R = \mathbb{Z} \otimes \mathbb{Z}$, and $S = m\mathbb{Z}^* \times 0$ $(m \in \mathbb{N})$). Let $J = (m + 1) \mathbb{Z} \times 0$. Evidently, $(R, \leq)$ is a partially ordered ring, and J is an ideal of R which is not convex.

(1) Put $a = (0, -1)$, $b = (-2, -2) \in R$. Then $((R \ltimes R; a, b), \leq_*)$ is a partially ordered ring by Theorem 2.1(2). Also, $0 \times J$ is a convex ideal, but $J \times J$ is not convex by Proposition 3.3(2) (though it contains the convex ideal $0 \times J$).

(2) Put $a = (0, 0)$, $b = (0, t) \in R$. Then $((R \ltimes R; a, b), \preceq)$ is a partially ordered ring by Theorem 2.1(5). Also, $0 \times R$ is obviously a convex ideal. While, $0 \times J$ is not convex by Proposition 3.3(5) (though it is contained in the convex ideal $0 \times R$).

We will give characterizations of the convexity of an ideal of $(R \ltimes R; a, b)$ with respect to the lexicographic set $L = (S_0 \times R) \cup (0 \times S)$ (as a generalization of Proposition 3.3(5)).

Hereafter, *we assume that an ideal I of $(R \ltimes R; a, b)$ is proper, unless otherwise stated. Also, we assume that $(R \times R; a, b)$ has the semi-cone L, that is, $(R \times R; a, b)$ is the partially ordered ring $((R \times R; a, b), \preceq)$.*

Theorem 3.5. *Let I be an ideal of $(R \ltimes R; a, b)$, and let $P_0 = p_1(I) \cap S_0$. Then the following hold.*

(1) For $P_0 = \emptyset$, I is convex in $(R \ltimes R; a, b)$ iff $p_2(I_0)$ is convex or $p_2(I_0) = R$, where $I_0 = I \cap (0 \times R)$. Specially, for $p_1(I) = 0$ (equivalently, $I = 0 \times p_2(I)$), I is convex iff $p_2(I)$ is convex or $p_2(I) = R$.

(2) For $P_0 \neq \emptyset$, I is convex in $(R \ltimes R; a, b)$ iff (i) $p_1(I)$ is convex, and (ii) $(P_0 \times R) \cup (0 \times S) \subset I$. It is possible to replace (ii) by (ii)' $I = p_1(I) \times R$.

Proof. For (1), note that $p_2(I_0)$ is an ideal of R. For the only if part, let $0 \leq y \leq y' \in p_2(I_0)$. Then $(0, 0) \preceq (0, y) \preceq (0, y') \in I$, thus $(0, y) \in I$, which yields $y \in p_2(I_0)$. For the if part, to see I is convex, let $(0, 0) \preceq (x, y) \preceq (x', y') \in I$. Then $x = x' = 0$ by $P_0 = \emptyset$. Thus $0 \leq y \leq y' \in p_2(I_0)$, which yields $y \in p_2(I_0)$. Hence $(x, y) = (0, y) \in I$.

For (2), for the only if part, to see (i), let $0 < x < y \in p_1(I)$. Then $(y, z) \in I$ for some $z \in R$. Thus $(0, 0) \prec (x, z) \prec (y, z) \in I$. Since I is convex, $(x, z) \in I$. Then $x \in p_1(I)$. Hence (i) holds, here $p_1(I) \neq R$ by (ii)' (as is seen below). We will show (ii)' holds. First, let $(p, y) \in P_0 \times R$. Then $(0, 0) \prec (p, y) \prec (2p, y') \in I$ for some $y' \in R$. Thus $(p, y) \in I$, hence $P_0 \times R \subset I$. Next, for $r \in R$, $(p, r), (p, 0) \in I$ for $p \in P_0$, thus $(0, r) = (p, r) - (p, 0) \in I$, which yields $0 \times R \subset I$. Finally, for $(x, r) \in p_1(I) \times R$, $(x, y) \in I$ for some $y \in R$. Thus, $(x, r) = (x, y) + (0, r - y) \in I$, which yields $p_1(I) \times R \subset I$. Hence (ii)' holds. For the if part, to see I is convex, let $(0, 0) \preceq (x, y) \preceq (x', y') \in I$. Then $0 \leq x \leq x' \in p_1(I)$. If $x = 0$, then $(x, y) = (0, y) \in 0 \times S$, thus $(x, y) \in I$ by (ii). If $x \in S_0$, $x \in P_0$ since $x \in p_1(I)$ by (i). Thus $(x, y) \in I$ by (ii). Therefore, $(x, y) \in I$. □

The following corollary holds by Theorem3.5 (noting that for $P_0 = \emptyset$, obviously $p_1(I)$ is convex).

Corollary 3.6. *For a convex ideal I of $(R \ltimes R; a, b)$, $p_1(I)$ is convex.*

Corollary 3.7. *For an ideal I of $(R \ltimes R; a, b)$, the following hold.*

(1) Let $p_1(I) \subset S \cup (-S)$ (in particular, let S be a cone). Then I is convex iff (i) $I = 0 \times p_2(I)$ such that $p_2(I)$ is convex or $p_2(I) = R$, or (ii) $I = p_1(I) \times R$ such that $p_1(I)$ is convex.

(2) Let $S \subset p_1(I)$. Then I is convex iff $L \subset I$ (equivalently, $S \times R \subset I$).

Remark 3.8. *In Theorem 3.5(1), $P_0 = \emptyset$ need not imply $p_1(I) = 0$. Also, in Corollary 3.7, the assumption in (1) or (2) is essential in its only if part.*

Indeed, let $I' = 0 \times R$ be an ideal, and $S' = S \times 0$ be a semi-cone in $R' = R \otimes R$ with $S_0S_0 \subset S_0$. Let $R'' = (R' \ltimes R'; a, b)$, but $a = (0, 0)$, $b = (0, r)$ with $r \in R$. Let $I = I' \times I'$, and $L = (S'_0 \times R') \cup (0 \times S')$. Then I is a convex ideal with respect to a semi-cone L in R''(by Theorem 2.1(5) and Proposition 3.3(5)), satisfying $P_0 = \emptyset$, but $p_1(I) \neq 0$ and $p_2(I) \neq R'$.

For R being an integral domain, $((R \ltimes R; a, b), \preceq) = (R \ltimes R, \preceq)$ by Corollary2.7(3)(b). From now on, we will deal with the partially ordered ring $R \ltimes R$ ($= (R \ltimes R; \preceq)$) under R being an integral domain.

Corollary 3.9. *Let R be an integral domain. For an ideal $A = (p, q) * (R \ltimes R)$, the following hold.*

(1) For $p = 0$, A is convex in $R \ltimes R$ iff qR is convex or $qR = R$.

(2) For $p \neq 0$, A is convex in $R \ltimes R$ iff $pR \cap S = 0$.

Proof. (1) holds by Theorem 3.5(1). For (2), for the only if part, suppose $pR \cap S \neq 0$. Then $A = p_1(A) \times R$ by Theorem 3.5(2), thus $(0, 1) \in A$. Hence, $(0, 1) = (p, q) * (x, y)$ for some $(x, y) \in R \ltimes R$, which yields $py = 1$. Then $p_1(A) = pR = R$, a contradiction. For the if part, A is convex by Theorem 3.5(1), because $p_2(A \cap (0 \times R)) = pR$ is convex by $pR \cap S = 0$. □

Let us define the following collection C in $R \ltimes R$ and its subcollections C_1, C_2, C_3 satisfying $C = C_1 \cup C_2 \cup C_3$.

$$C = \{I \mid I \text{ is a non-zero convex ideal of } (R \ltimes R, \preceq)\},$$
$$C_1 = \{I \in C \mid p_1(I) = 0\},$$
$$C_2 = \{I \in C \mid p_1(I) \cap S \neq 0\},$$
$$C_3 = \{I \in C \mid p_1(I) \neq 0,\ p_1(I) \cap S = 0\}.$$

The following proposition follows from Theorem 3.5.

Proposition 3.10. *Let R be an integral domain with a semi-cone S. Let $I = (p, t) * (R \times 0) + (0, q) * (R \times 0)$ be a non-zero ideal of $R \ltimes R$. Then the following hold.*

(1) $I \in C_1$ iff $I = 0 \times (tR + qR)$ such that $tR + qR$ is a non-zero convex or $tR + qR = R$.

(2) $I \in C_2$ iff $I = pR \times R$ such that pR is convex with $pR \cap S \neq 0$.

(3) $I \in C_3$ iff $I = (p, t) * (R \times 0) + (0, q) * (R \times 0)$ such that $p \neq 0$, $pR \cap S = 0$, and qR is convex or $qR = R$.

For elements p, q in R, the symbol $p \mid q$ means $q = rp$ for some $r \in R$. We recall the following fact in [5].

Fact 3.11. *For a principal ideal domain R, a subset A of a ring $(R \ltimes R; a, b)$ is an ideal (possibly, $A = R \times R$) iff $A =$*

$(p, t) * (R \times 0) + (0, q) * (R \times 0)$ *for some* $p, t, q \in R$ *satisfying (i)* $p \mid at$, *(ii)* $p \mid aq$, *and (iii)* $pq \mid (p^2 + pbt - at^2)$. *In particular, we have*

Lemma 3.12. *For a principal ideal domain* R, *a subset* A *of* $R \ltimes R$ *is an ideal (possibly,* $A = R \times R$*) iff* $A = (p, t) * (R \times 0) + (0, q) * (R \times 0)$ *for some* $p, t, q \in R$ *satisfying* $q \mid p$. *(Obviously, for* $A \subset 0 \times R$, A *is an ideal iff* $A = (0, q') * (R \times 0)$ *for some* $q' \in R$).

For a principal ideal domain R, we obtain the following characterization for the convexity of ideals in $R \ltimes R$ in view of Proposition 3.10 and Lemma 3.12.

Theorem 3.13. *Let* R *be a principal ideal domain with a semi-cone* S. *Then the collection* C *in* $R \ltimes R$ *is the union of the following collections* C_1, C_2, *and* C_3.

$C_1 = \{0 \times qR \mid q \in R, qR \text{ is non-zero convex or } qR = R\}$,

$C_2 = \{pR \times R \mid p \in R, pR \cap S \neq 0, \text{ and } pR \text{ is convex}\}$,

$C_3 = \{(p, t) * (R \times 0) + (0, q) * (R \times 0) \mid p, q, t \in R, q \mid p, p \neq 0, pR \cap S = 0, \text{ and } qR \text{ is convex or } qR = R\}$.

In Theorem 3.13, for S being a cone in R, $C_3 = \emptyset$ (by $pR \cap S \neq 0$ for any $p \neq 0$). Hence we have the following.

Corollary 3.14. *Let* R *be a principal ideal domain with a cone* S. *Then* $C = \{pR \times R, 0 \times qR \mid p, q \in R\ (q \neq 0), pR \text{ and } qR$ are convex in $R\}$.

We will apply Theorem 3.13 to $R = \mathbb{Z}$. Let us recall the following fact in [3].

Fact 3.15. *Every (non-zero) semi-cone* S *of* $\mathbb{Z}$ *is precisely the set* $d_1\mathbb{Z}^* + \cdots + d_r\mathbb{Z}^*$ *for some* $d_1, d_2, \ldots, d_r$ *in* S *with* $0 < d_1 < \cdots < d_r$. *Define* $d_S = \gcd(d_1, \ldots, d_r)$. *Then the positive integer* d_S *is uniquely determined by the semi-cone* S *(indeed, for the ideal* I *of* $\mathbb{Z}$ *generated by* S, $I = \sum_{i=1}^{r} d_i\mathbb{Z}$, *hence* $I = d_S\mathbb{Z}$).

Corollary 3.16. *Let* S *be a (non-zero) semi-cone of* $\mathbb{Z}$. *Then the non-zero convex ideals of* $(\mathbb{Z} \ltimes \mathbb{Z}, \preceq)$ *are precisely the sets* $0 \times n\mathbb{Z}$ $(1 \leq n, n \mid d_S)$, *or* $m\mathbb{Z} \times \mathbb{Z}$ $(2 \leq m, m \mid d_S)$. *Hence, the number of convex ideals of* $(\mathbb{Z} \ltimes \mathbb{Z}, \preceq)$ *is equal to twice that of positive divisors of the number* d_S.

Proof. Every $I \in C$ can be written as $I = (m, t) * (\mathbb{Z} \times 0) + (0, n) * (\mathbb{Z} \times 0)$ for some $m, n \in \mathbb{Z}^*$ and $t \in \mathbb{Z}$ by Lemma 3.12. Obviously, $C_3 = \emptyset$ by $m\mathbb{Z} \cap S \supset mS \neq 0$ for any $m \neq 0$. Note that for a non-zero ideal $J = j\mathbb{Z}$ $(j > 0)$ of $\mathbb{Z}$, J is convex or $J = \mathbb{Z}$ *iff* $J \supset S$, that is, $j \mid d_S$ ([3, Proposition 3.4]). Therefore, the corollary follows from Theorem 3.13. □

References

[1] L. Gillman and M. Jerison, Rings of continuous functions, Van Nostrand Reinhold company, 1960.

[2] Y. Kitamura and Y. Tanaka, Ordered rings and order-preservation, Bull. Tokyo Gakugei Univ., Nat. Sci., **64** (2012), 5–13.

[3] Y. Kitamura and Y. Tanaka, Partially ordered rings, Tsukuba J. Math., **38** (2014), 39–58.

[4] Y. Kitamura and Y. Tanaka, Partially ordered rings II, Tsukuba J. Math., **39** (2015), 181–198.

[5] Y. Kitamura and Y. Tanaka, Product extensions of commutative rings, Bull. Tokyo Gakugei Univ., Nat. Sci., **67** (2015), 1–8.

[6] Y. Kitamura and Y. Tanaka, Finitely generated semi-cones in product rings, to appear

[7] T. Y. Lam, A first course in noncommutative rings, Graduate Texts in Mathematics, **131**, Springer, 1991.

積拡大環と半順序環

北　村　　　好*・田　中　祥　雄*

数学分野

要　旨

論文［3］(又は，［2］）において，可換環Rの拡大環として，積$R \times R$に積拡大環なる概念が導入され，その代数的構造が考察された（[3]）。環の順序構造に関しては，正のコーン（positive cone）の一般化として，非負のコーン（non-negative semi-cone）なる概念が導入され（[1]），“非負の半コーン”を短縮して，“半コーン”（semi-cone）と呼んだ（[2]）。半コーンは環の半順序を決定する。[2] において，基本的な積環における順序構造が標準的な半コーンによって考察された。本稿では，ある種の半コーンによって，積拡大環の順序構造を考察する。すなわち，積拡大環がこれらの半コーンに関して，半順序となるための特徴づけを与える。さらに，そこにおける凸イデアル（convex ideal）を考察する。特に，辞書式順序をもつ積拡大環のイデアルが凸イデアルになるための特徴づけを与える。その応用として，整数環のそれ自身による自明な拡大（trivial extension）において，辞書式順序に関する凸イデアルとその個数を厳密に決定する。

キーワード: 積拡大環，直積環，半順序環，半コーン，凸イデアル，辞書式順序

参考文献

[1] Y. Kitamura and Y. Tanaka, Partially ordered rings, Tsukuba J. Math., **38** (2014), 39–58.

[2] Y. Kitamura and Y. Tanaka, Partially ordered rings II, Tsukuba J. Math., **39** (2015), 181–198.

[3] Y. Kitamura and Y. Tanaka, Product extensions of commutative rings, Bull. Tokyo Gakugei Univ., Nat. Sci., **67** (2015), 1–8.

＊ 東京学芸大学（184-8501　小金井市貫井北町 4-1-1）