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Ordered fields and metrizability

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Abstract

We recall that an ordered field is a field which has a linear order and the order topology by the order. Ordered fields have played important roles in the theory of the real number field **R** in terms of Archimedes' axiom or the axiom of continuity. Ordered fields give algebraic and topological principles in *Analysis*, *Algebra*, etc. with respect to the structure of the field **R**.

In this paper, we give metrization theorems on ordered fields, and examples on non-Archimedean ordered fields, etc. Also, as materials around ordered fields, we consider metrizability of ordered (additive) groups, and definitions of real number fields.

Key words: ordered field, real number field, Archimedes' axiom, axiom of continuity, metrizability, metric space, Lindelöf space

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Introduction

Let **R** (resp. Q) be the real (resp. rational) number field, and let N be the set of all natural numbers.

Ordered fields have played important roles in the theory of the field **R** in terms of Archimedes' axiom or the axiom of continuity. They give algebraic and topological principles in *Analysis* (differential and integral calculus) or *Algebra*, etc., with respect to the structure and properties of the field **R**. For (teaching) materials for the continuity of **R**, see [8] (or [9]).

As a generalization of Euclidean n-spaces (in particular, **R**), metric spaces are one of the most important materials in *General Topology* (or *Set-theoretic Topology*). A topological space X is *metrizable* (or simply, *metric*) if its topology coincides with the topology determined by some metric function from $X \times X$ to **R**. The problem on metrizability of topological spaces is a very important subject in *General Topology*.

We note that every Archimedean ordered field is (separable) metrizable, but not every ordered field is metrizable; see [1] (or Example 2 later).

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In [1], it is shown that for an ordered field K with canonical uniformity $\mathcal{U}$, K is metrizable $\Leftrightarrow$ $\mathcal{U}$ has a countable base $\Leftrightarrow$ K^+ ($=\{x \in K : 0 < x\}$) contains a countable set A such that for each $x \in K^+$, there exists $a \in A$ with $0 < a < x$. For uniformities or uniform spaces, see [5] (or [2], [3]).

We show the following metrization theorem on ordered fields holds, and we will expand this theorem to (additive) ordered groups.

Theorem: Let K be an ordered field. Then the following are equivalent.

(a) K is metrizable.

(b) K is a first countable space (generally, a k−space).

(c) K contains a non−discrete countable subspace.

Also, we show that for an ordered fielld K, K is seprable and metrizable $\Leftrightarrow$ K is separable $\Leftrightarrow$ K is Lindelöf.

We give two examples on non−Archimedean ordered fields (one is separable metrizable, another is not metrizable). Finally, as materials around ordered fields, we consider metrizability of ordered (additive) groups, and the definitions of real numbers.

We conclude this introduction by recording some main definitions used in this paper.

We recall that a *field* is a commutative ring with the unity element 1, and each element of $K-\{0\}$ has a multiplicative inverse.

A field $(K, \leq)$ having a linear order (or total order) $\leq$ is an *ordered field* if K has the following (i) and (ii).

(i) If $x < y$, then $x + z < y + z$. If $x < y$ and $0 < z$, then $xz < yz$.

(ii) K is a topological space having the order topology (or open−interval topology); that is, $G \subset K$ is open in K iff for each $x \in G$, there exists an open interval (a, b) ($=\{y \in K : a < y < b\}$) such that $x \in (a, b) \subset G$; equivalently, there exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset G$.

We note that every ordered field K contains (isomorphically) the rational number field Q, but K contain no isolated points.

Let $(K, \leq)$ be an ordered field. For $A \subset K$, a *subspace* A in K is a topological space having the usual relative topology determined by the topology in K. We note that the relative topology on A need not coincide with the order topology on A by the linear order restricted to A from the order $\leq$ on K (their topologies are coincide on compact or connected subsets, etc.).

Let $(K, \leq)$ be an ordered field. Then K is *Archimedean* if it satisfies the following Archimedes' axiom: For each $\alpha, \beta \in K$ with $0 < \alpha < \beta$, there exists $n \in N$ such that $\beta < n\alpha$. In other words, K is Archimedean iff for each $x \in K$, there exists $n \in N$ with $x < n$; thus, K is non−Archimedean iff there exists $a \in K$ such that $a > n$ for all $n \in N$.

We recall the axiom of continuity which is stronger than Archimedes' axiom. For an ordered field $(K, \leq)$, $(A|B)$ is a (Dedekind) *cut* in K if $K = A \cup B$, $A \cap B = \emptyset$ with $A, B \neq \emptyset$, and for any $x \in A, y \in B, x < y$.

An ordered field $(K, \leq)$ satisfies the *axiom of continuity* if for each cut $(A|B)$ in K, K has *max* A or *min* B, but doesn't have both of *max* A and *min* B. We say that K is *Dedekind−complete* if it satisfies the axiom of continuity. The rational number field Q is Archimedean, but not Dedekind−complete. The real number field $\mathbf{R}$ is Dedekind−complete.

Let $(K, \leq)$ be an ordered feld. For $x \in K$, define the absolute value $|x| \in K$ by $|x| = x$ if $x \geq 0$, and $|x| = -x$ if $x < 0$. Then, for all x, $y \in K$, $|x| = |-x|$, and $|x + y| \leq |x| + |y|$.

It is easy to show that every ordered field is a normal space (i.e., for every disjoint closed sets F_i (i=1,2), there exist disjoint open sets G_i containing F_i), using the triangle inequality of the absolute value $|x|$. But, every ordered field K need not be metrizable; see [1; Remark 2.7] (or Example 2 later).

Let X be a (topological) space. For a subset A of X, $a \in X$ is an *accumulation point* of A if any nbd (i.e., open neighborhood) of

a meets $A-\{a\}$ (i.e., $a \in cl(A-\{a\})$). A space X is *compact* (resp. *Lindelöf*) if every open cover of X has a finite (resp. countable) subcover. A space X is *first countable* (resp. *second countable*) if each point of X has a countable nbd (or local) base (resp. X has a countable base). A space X is a *k–space* if $F \subset X$ is closed iff $F \cap C$ is closed in C for every compact subset C of X. Every locally compact space or first–countable space is a *k–space*. A space X is *separable* if it has a countable dense subset D (i.e., $clD = X$). For other topological terminologies used in this paper, see [2], [3], for example.

Every metric space is obviously first countable. As is well–known, for a metric space X, X is second countable $\Leftrightarrow$ X is separable $\Leftrightarrow$ X is Lindelöf.

A space $(X, \leq)$ with a linear order $\leq$ is called a *linearly ordered topological space* (simply, *LOTS*) if X has the order topology. For LOTS, see [6] (or [3], [7]), etc. An ordered field K is a LOTS which has the usual four arithmetic operations and order relation. In LOTS, "cuts" are similarly defined. For basic topological properties of LOTS in terms of the "cuts", see [7] (or [3]).

Results

Several equivalent conditions to the axiom of continuity or Archimedes' axiom are given in [8], [9], etc. First, let us review some equivalent conditions to Archimedes' axiom. In the following theorem, the equivalences among (a), (b), and (c) are well–known (see [8], [9], etc.), and so is (a) $\Leftrightarrow$ (d) (see [4; 0.21], etc.). (c) $\Leftrightarrow$ (f) is routinely shown. (a) $\Leftrightarrow$ (e) is due to [9].

Theorem 1. For an ordered field K, the following are equivalent.

(a) K is Archimedean.

(b) The sequence $\{1/n : n \in N\}$ converges to $0 \in K$.

(c) Q is a dense subset of K.

(d) K is isomorphic to a subfield of $\mathbf{R}$.

(e) The set $\{1/n : n \in N\} \cup \{0\}$ is compact in K.

(f) K has a countable base each of whose elements meets Q.

Remark 1. Let us recall some equivalent conditions (which are well–known) to the axiom of continuity (i.e., the Dedekind–completeness).

For an ordered field K, the following are equivalent.

(a) K is Dedekind–complete.

(b) Every lower (resp. upper) bounded subset of K has an infimum (resp. supremum) in K.

(c) Every lower bounded decreasing (or upper bounded increasing) sequence has a limit point in K.

(d) Every bounded infinite subset of K has an accumulation point in K.

(e) K is isomorphic to $\mathbf{R}$.

(f) The interval [0, 1] (or any closed interval [a, b]) is compact in K.

(g) The interval [0, 1] (or any interval A; that is, for any $a, b \in A$ $(a < b)$, $[a, b] \subset A$) is connected in K.

(h) K is Archimedean, and every Cauchy sequence has a limit point in K. Here, a sequence $\{a_n : n \in N\}$ is *Cauchy* if for each $\epsilon > 0$, there exists $k \in N$ such that $|a_m - a_n| < \epsilon$ if $m, n > k$.

In Theorem 1, if K is Archimedean, then $K - Q$ $(\neq \emptyset)$ is also dense in K by (b) and (c). But, by the following example, it is impossible to replace "Q" by "$K - Q$" in (c), and the latter part in (f) is essential.

Example 1. A second countable (hence, separable metrizable), non–Archimedean ordered field $Q(x)$ which contains a dense subset $Q(x) - Q$.

Proof. Let $Q(x)$ be the field of all rational functions in the variable x with coefficients in Q. For $\eta(x) = \pm\dfrac{f(x)}{g(x)} \in Q(x)$,

$f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j$ with a_m, $b_n > 0$, define $\eta(x) > 0$ if the sign of the fraction is "+", and $\eta(x) < 0$ if " − ". For $\eta(x)$, $\xi(x) \in Q(x)$, define $\eta(x) \leq \xi(x)$ if $0 \leq \xi(x) - \eta(x)$. Then $(Q(x), \leq)$ is a countable ordered field (for example, for $\eta_1(x) \leq \eta_2(x)$ and $\eta_2(x) \leq \eta_3(x)$, $\eta_1(x) \leq \eta_3(x)$ by $\eta_3(x) - \eta_1(x) = (\eta_3(x) - \eta_2(x)) + (\eta_2(x) - \eta_1(x)) > 0)$). Since $Q(x)$ is countable, ovbiously it has a countable base, thus, $Q(x)$ is metrizable (see Corollary 2 later). $Q(x)$ is not Archimedean since $n < x$ for all $n \in N$. Clearly, Q is not dense in $Q(x)$, but $Q(x) - Q$ is dense in $Q(x)$. To see this, let $\eta(x) < \xi(x)$ (containing a case $\eta(x) \equiv a$, or $\xi(x) \equiv b$). Then $\eta(x) < \frac{1}{2}(\eta(x) + \xi(x)) < \xi(x)$. Considering a case where $\frac{1}{2}(\eta(x) + \xi(x)) \in Q$, it would suffices to consider a case where $\eta(x) \equiv a$, $\xi(x) \equiv b$. Let $c = \frac{1}{2}(a + b) \in Q$ (if $c = 0$, put $c = \frac{1}{2}b$), and $\mu(x) = \frac{cx}{x+1} \in Q(x) - Q$. Then $a < \mu(x) < b$. Hence, $Q(x) - Q$ is dense in $Q(x)$.

A family $\mathcal{A}$ of sets in a space X is *locally finite* (resp. *discrete*) if each point of X has a nbd which meets at most finite many elements (resp. one element) of $\mathcal{A}$.

Let us recall the following Nagata−Smirnov Metrization theorem; see [3; Theorem 4.4.7], etc.

Nagata-Smirnov Metrization: A regular space is merizable iff it has a base which is a countable union of locally finite families.

This Metrization theorem remains true for every ordered field K in view of the proof of [2; Theorem 9.1(p.194)] (for the "if" part, K is homeomorpically embedded in the Hilbert (metric) space in [2; p.191]. Note that for real valued functions defined on K into **R**, the axiom of continuity of **R** holds in their ranges, thus, Theorem 1, Remark 1, etc. can be used there).

A space X is *collectionwise normal* if for every discrete collection $\{F_\alpha : \alpha \in A\}$ of closed subsets of X, there exists a discrete collection $\{G_\alpha : \alpha \in A\}$ of open subsets such that $F_\alpha \subset G_\alpha$ for every $\alpha \in A$. As is well-known, every LOTS is collectionwise normal; see [7], etc.

A sequence $\{\mathcal{G}_n : n \in N\}$ of open covers of a space X is a *development* for X if each point x of X has a countable nbd base $\{St(x, \mathcal{G}_n): n \in N\}$, where $St(x, \mathcal{G}_n) = \bigcup\{G \in \mathcal{G}_n : x \in G\}$. (A space X with a development is *developable*, and X is a *Moore space* if X moreover regular; see [3], [7], etc.)

We recall the following Bing Metrization theorem; see [3; Theorem 5.4.1] (which is shown by use of the Nagata−Smirnov Metrization theorem).

Bing Metrization theorem: A space is metrizable iff it is collectionwise normal and has a development.

Theorem 2. For an ordered field K, the following are equivalent.

(a) K is metrizable.

(b) K contains an infinite convergent sequence.

(c) K is a k−space (in particular, first countable space).

(d) K contains an infinite compact subspace.

(e) K contains a non−discrete countable subspace.

Proof. For (a) ⇒ (b), K has no isolated points. Thus, for (any) $x \in K$, $x \in cl(K - \{x\})$. Since K is first countable, there exists a sequence L in $K - \{x\}$ converging to the point x. Thus, K contains the infinite convergent sequence L. For (b) ⇒ (d), any convergent sequence containing its limit point is compact in K. For (a) ⇒ (c), let $A \subset K$ be not closed in K. Then there exists a sequence L in A converging to a point $x \notin A$. Let $C = L \cup \{x\}$. Then C is compact in K, but $A \cap C$ is not closed in C. This shows that K is a k-space. For (c) ⇒ (d), let $x \in K$. Since $K - \{x\}$ is not closed in K, $(K - \{x\}) \cap C$ is not closed in C for some compact subset C of K. Thus C is an infinite compact subset of K. For (d) ⇒ (e), the infinite compact set in (d) has a countable subset A having an accumulation point in A. Then A is a non−disctere countable subspace in K. For (e) ⇒ (a), K is a LOTS, then as is well−known, K is collectionwise normal. Thus, it suffices to show that K has a development by the Bing Metrization theorem. Let A be a non-discrete countable subspace in K. Then A has an accumulation point α in A (hence, $\alpha \in cl(A - \{\alpha\})$ in K). Let $A - \{\alpha\} = \{\alpha_n : n \in$

$N\}$. For each $n \in N$, let $\epsilon_n = |\alpha - \alpha_n| > 0$. Then, for each $\epsilon > 0$, there exists $\epsilon_n < \epsilon$, because $(\alpha - \epsilon, \alpha + \epsilon) \cap (A - \{\alpha\}) \neq \emptyset$. For each $n \in N$, let $\mathcal{G}_n = \{(x - \epsilon_n, x + \epsilon_n): x \in K\}$. Then, the sequence $\{\mathcal{G}_n : n \in N\}$ of open covers of K is a development of K. Indeed, let $x \in K$. Suppose that for some nbd $V(x) = (x - \epsilon, x + \epsilon)$, any $St(x, \mathcal{G}_n)$ is not contained in $V(x)$. Then, we can choose a sequence $\{x_n : n \in N\}$ such that $x_n \in St(x, \mathcal{G}_n) - V(x)$. Take $\epsilon_i < \epsilon$, and $\epsilon_j < \epsilon - \epsilon_i$, and let $\epsilon_k = min\ \{\epsilon_i, \epsilon_j\}$. Since $x_k \in St(x, \mathcal{G}_k)$, take any element $(a - \epsilon_k, a + \epsilon_k)$ in $\mathcal{G}_k$ which contains the points x_k, x. Since $|x_k - a|, |a - x| < \epsilon_k$, $|x_k - x| < 2\epsilon_k < \epsilon$, then $|x_k - x| < \epsilon$. Thus, $x_k \in V(x)$. This is a contradiction. Hence, the sequence $\{\mathcal{G}_n : n \in N\}$ is a development for K.

Remark 2. A space is *countably compact* if every countable open cover has a finite subcover (equivalently, every infinite subset has an accumulation point). Let K be an ordered field. Then every countably compact subset of K is compact and metrizable in view of Theorem 2.

For a LOTS X, if X is separable, X is hereditarily Lindelöf (see [6; p.8]), thus every countably compact subset of X is compact. But, every countably compact LOTS need not be compact. Also, every compact LOTS need not be metrizable even if it is hereditarily separable, hereditarily Lindelöf, and first countable (*two arrows space* by the lexicographic order); see [3; 3.10.C].

A point x in a space X is a $G_\delta - set$ if there exists a sequence $\{G_n : n \in N\}$ of open subsets such that $\{x\} = \bigcap\{G_n : n \in N\}$.

Corollary 1. For an ordered field K, the following are equivalent.

(a) K is metrizable.

(b) Some point of K has a countable nbd base.

(c) Some point of K is a G_δ–set.

Proof. (a) ⇒ (b) ⇒ (c) is obvious. For (c) ⇒ (a), let $x \in K$ be a G_δ–set. Then there exists a sequence $\{(a_n, b_n): n \in N\}$ of open intervals such that $\{x\} = \bigcap\{(a_n, b_n): n \in N\}$. Let $A = \{a_n, b_n : n \in N\}$. Since K has no isolated points, the point x is not isolated in K. Then x is an accumulation point of A in K (indeed, suppose not. Take a nbd $V(x)$ of x with $V(x) \cap A = \emptyset$, and a point $p \in (V(x) - \{x\})$. Thus, $p \in \bigcap\{(a_n, b_n): n \in N\}$, a contradiction). Thus K is metrizable by Theorem 2.

As is well-known, every regular space is separable and metrizable iff it has a countable base (Urysohn or Tychonoff). We recall that, for a metrizable space X, X has a countable base ⇔ X is separable ⇔ X is Lindelöf.

We note that even if an ordered field K is metrizable, K need not be separable (hence, neither second countable nor Lindelöf); see [1].

Corollary 2. For an ordered field K, the following are equivalent. In particular, an Archimedean ordered field is separable and metrizable ([1]).

(a) K is separable and metrizable.

(b) K is second countable.

(c) K is separable.

(d) K is Lindelöf.

Proof. We show that (c) or (d) implies (a). For (c) ⇒ (a), let D be a countable dense subset of K. If $K = D$, then each point of K is obviously a G_δ–set, thus K is metrizable by Corollary 1. If $K \neq D$, then a point $x \notin D$ is an accumulation point of D. Thus, K is metrizable by Theorem 2. For (d) ⇒ (a), let us show that K is first countable, then K is metrizable by Corollary 1, and thus separable. Suppose that K is not first countable. Since K is Lindelöf, and has an open cover $\mathcal{V} = \{(-\infty, x): x > 0\}$, where $(-\infty, x) = \{y \in K : y < x\}$, then $\mathcal{V}$ has a countable subcover $\{(-\infty, x_n): n \in N\}$. Let $A = \{x_n : n \in N\}$. Since K is not first countable, the sequence $L = \{1/x_n : x_n \in A\}$ has no accumulation points in K by Theorem 2, thus, $0 \in K$ is not an accumulation point of L. Then, since $0 < 1/x_n$ $(n \in N)$, there exists $a \in K$ such that $0 < a < 1/x_n$ $(n \in N)$; namely, $x_n < 1/a$ for all $n \in N$. Thus $1/a \notin K$. This is a contaradiction. For the latter part of the corollary, every Archimedean ordered field is separable by Theorem

1, then it is metrizable.

A space X is *locally separable* if each point of X has a separable nbd. For a space X and a pairwise disjoint open cover of X, X is the *topological sum* of the cover. For the following corollary, since K is locally separable, K is metrizable in view of the proof of Corollary 2. Thus, K is the topological sum of separable and metrizable subspaces; see [3; 4.4.F]).

Corollary 3. If K is a locally separable ordered field, then K is the topological sum of separable and metrizable subspaces.

In terms of property (e) in Theorem 2, let us give characterizations for ordered fields to be Archimedean or Dedekind−complete, using subsets of Q.

Theorem 3. Let K be an ordered field. Then the following hold.
(1) K is metrizable iff it contains a countable subset having an accumulation point.
(2) K is Archimedean iff it contains a subset of Q having an accumulation point in K.
(3) K is Dedekind−complete iff every bounded infinite subset of Q has an accumulation point in K.

Proof. (1) is a restatement of (a) ⇔ (e) in Theorem 2. For (2) and (3), their "only if" parts holds by Theorem 1 and Remark 1, then we show that their "if" parts hold. For (2), let $A = \{r_n : n \in N\} \subset Q$ have an accumulation point $a \in K$, here we can assume that the points r_n ($n \in N$) and a are distinct. For each $m, n \in N$, let $\varepsilon_{mn} = |r_m - r_n|$. Let $\varepsilon > 0$. Then for some distinct points r_m, r_n in A, $|r_m - a|, |r_n - a| < \varepsilon/2$, thus $0 < \varepsilon_{mn} < \varepsilon$. But, since $r_m, r_n \in Q$, $1/k \le \varepsilon_{mn}$ for some $k \in N$. Hence $1/k < \varepsilon$. This shows that K is Archimedean. For (3), let every bounded infinite subset of Q have an accumulation point in K. Then, every lower bounded decreasing sequence in Q has an accumulation point (which is also the limit point) in K. Thus the sequence $\{1/n : n \in N\}$ converges to $0 \in K$, hence Q is dense in K by Theorem 1. Therefore, every lower bounded decreasing sequence $\{x_n : n \in N\}$ in K has a limit point in K. Indeed, assuming the points x_n are distinct, for each $n \in N$, take $r_n \in Q$ with $x_{n+1} \le r_n \le x_n$. Then the sequence $\{r_n : n \in N\}$ in Q is a lower bounded decreasing sequence, thus it has a limit point. Then the sequence $\{x_n : n \in N\}$ has the same limit point. Hence K is Dedekind-complete by Remark 1.

Corollary 4. Let K be an ordered field. Then the following hold.
(1) K is metrizable iff it contains an infinite Cauchy sequence.
(2) K is Archimedean iff it contains an infinite Cauchy sequence in Q.
(3) K is Dedekind−complete iff it contains an infinite Cauchy sequence in Q, and every Cauchy sequence in Q has a limit point in K.

Proof. The "only if" parts of (1), (2), and (3) hold by Theorems 1, 2, and Remark 1, for every convergent sequence is Cauchy. For the "if" parts of (1) and (2), let $\{a_n : n \in N\}$ be a Cauchy sequence with the a_n distinct. Then the sequence $\{|a_n - a_{n+1}| : n \in N\}$ in K has an accumulation point $0 \in K$. Thus, the parts hold by Theorem 3. For the "if" part of (3), let A be a bounded infinite subset of Q. Since K is Archimedean by (2), we can take a decreasing sequence $\{[a_n, b_n] : n \in N\}$ in K such that each $[a_n, b_n] \cap A$ is infinite, and the decreasing sequence $\{b_n - a_n : n \in N\}$ converges to $0 \in K$. Choose an infinite sequence $L = \{r_n : n \in N\}$ in A with $r_n \in [a_n, b_n]$, then L is a Cauchy sequence in Q. Since L has a limit point in K, A has an accumulation point in K. Hence, K is Dedekind−complete by Theorem 3.

Remark 3. Related to Remark 1, as additions to (3) in Theorem 3 and Corollary 4, similarly the following equivalences hold: For an ordered field K, K is Dedekind-complete ⇔ every lower bounded decreasing (or upper bounded increasing) sequence in Q has a limit point in K ⇔ every lower (resp. upper) bounded subset of Q has an infimum (resp. supremum) in K. Also, K is Dedekind−complete iff K has a decreasing sequence $\{[a_n, b_n] : n \in N\}$ ($a_n, b_n \in Q$, $a_n \neq b_n$) with $(b_n - a_n) \to 0$, and every such sequence $\{[x_n, y_n] : n \in N\}$ has a point belonging to all $[x_n, y_n]$.

Now, a non−empty collection $\mathcal{F}$ of subsets of X is called a *filter* if (i) $\emptyset \notin \mathcal{F}$; (ii) if $A \in \mathcal{F}$ and $A \subset B$, then $B \in \mathcal{F}$; and (iii) if A, $B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$. For an commutative ring A with 1, a subring I in A with $I \neq A$ is an *ideal* if for any $a \in I$ and $x \in A$, $xa \in I$. An *ultrafilter* is maximal with respect to set inclusion, and so is a *maximal ideal*.

Example 2. A non-Archimedean ordered field which is not first countable (hence, not metrizable), nor Lindelöf.

Proof. Let R^N be the set of all functions from N to $\mathbf{R}$. Then R^N is a commutative ring with 1 (for $f, g \in R^N$, let $(f+g)(n) = f(n) + g(n)$, and $(fg)(n) = f(n)g(n)$). Let $\mathcal{P}$ be an ultrafilter on N with $\cap\, \mathcal{P} = \emptyset$ (hence, $\mathcal{P}$ contains no finite sets). Let $I_{\mathcal{P}} = \{f \in R^N : f^{-1}(0) \in \mathcal{P}\}$. Then $I_{\mathcal{P}}$ is a maximal ideal in R^N. Let $K_{\mathcal{P}} = R^N/I_{\mathcal{P}}$ be the residue class (for $[f], [g] \in K_{\mathcal{P}}$, $[f] = [g]$ iff $\{n \in N : f(n) = g(n)\} \in \mathcal{P}$). Since $I_{\mathcal{P}}$ is a maximal ideal, $K_{\mathcal{P}}$ is a field, as is well-known. Define $[f] < [g]$ iff $\{n \in N : f(n) < g(n)\} \in \mathcal{P}$. Since $\mathcal{P}$ is an ultrafilter, $\leq$ is a linear order in $K_{\mathcal{P}}$. Thus, $(K_{\mathcal{P}}, \leq)$ is an ordered field. $K_{\mathcal{P}}$ is not Archimedean. Indeed, define let $h(i) = i$, $n(i) = n$ $(i \in N)$. Then, for all $n \in N$, $n = [n] < [h]$. $K_{\mathcal{P}}$ is not first countable. To see this, we show that $K_{\mathcal{P}}$ has no infinite sequence (with positive terms) converging to a non-isolated point $[0]$. Thus, let us show that for any countable set $\{[f_n] : n \in N\} \subset K_{\mathcal{P}}$ with $[f_n] > [0]$, there exists $[g] \in K_{\mathcal{P}}$ such that $[0] < [g] < [f_n]$ $(n \in N)$. For each $n \in N$, let $A_n = \{i \geq n : f_n(i) > 0\}$, and $B_n = A_1 \cap A_2 \cap \cdots \cap A_n$. Then $A_n \in \mathcal{P}$ by $\cap\, \mathcal{P} = \emptyset$. Thus, each $B_n \in \mathcal{P}$ and $\bigcap\{B_n : n \in N\} = \emptyset$. Define $g \in R^N$ by $g(i) = 1$ if $i \in N - B_1$, and let $0 < g(i) < min\{f_m(i) : m \leq n\}$ if $i \in B_n - B_{n+1}$. Then $[0] < [g] < [f_n]$ for all $n \in N$. Then $K_{\mathcal{P}}$ is not first countable, hence not metrizable. Thus $K_{\mathcal{P}}$ is not Lindelöf by Corollary 2.

Remark 4. By the above proof, $Q^N/I_{\mathcal{P}}$ is also a non−Archimedean ordered field which is neither first countable nor Lindelöf. Whereas, for the set Z of integers, $Z^N/I_{\mathcal{P}}$ is not a field, but it is an ordered ring (integral domain) which is a uncoutable discrete space (hence, neither separable nor Lindelöf).

Remark 5. Let $C(X)$ be the set of all continuous map from a completely regular space X to $\mathbf{R}$. Let M be a maximal ideal in the commutative ring $C(X)$, and let $C(X)/M$ be the residue class ordered field. (When $C(X)/M$ is isomorphic to $\mathbf{R}$, such a maximal ideal M is called *real*, and its characterization is given in [4; Theorem 5.14]). For an ordered field $K = C(X)/M$, K is isomorphic to $\mathbf{R}$ if K satisfies one of the following properties:

(a) K is Archimedean; (b) K is first countable; (c) Some point of K is a G_δ−set; (d) K is a k−space; (e) K is locally separable; (f) K is Lindelöf.

Indeed, the result for case (b) holds in view of [4; Theorem 13.8] and the proof of Example 2. Thus, the result for the rest cases holds by Thorem 2, and Corollaries 1, 2, and 3.

Finally, as materials around ordered fields, let us give metrizability of ordered (additive) groups, and the definitions of real number fields.

(*Ordered groups*): For an additive group G (which is commutative), $(G, \leq)$ is an *ordered group* if $\leq$ is a linear order, and G has the order topology by $\leq$. Every ordered ring is an ordered group. Let $(G, \leq)$ be an ordered additive group. For $a, b \in G$, let $a < b$ iff $0 < b - a$ (equivalently, $a < b$ iff $a + x < b + x$ for $x \in G$). For $x \in G$, define the absolute value $|x| \in G$ by $|x| = x$ if $x \geq 0$, and $|x| = -x$ if $x < 0$. Thus, for $x, y \in G$, $|x + y| \leq |x| + |y|$. Also, a subset $A \subset G$ is open in G iff for each $a \in A$, there exists $\epsilon > 0$ such that $(a - \epsilon, a + \epsilon) \subset A$. It is shown that G is normal by use of the value $|x|$; actually, G is collectiowise normal since G is a LOTS. Also, some point $a \in G$ is an isolated point; G_δ−set in G iff so is any point X of G respectively (by $y \in (x - \epsilon, x + \epsilon)$ iff $a + (y - x) \in (a - \epsilon, a + \epsilon)$).

The following hold in view of the proofs of Theorem 2, and Corollaries 1, 2, and 4. Related to the corollary below, the author doesn't know whether G is (separable) metrizable if G is Lindelöf.

Theorem: For a non-discrete, ordered additive group G, the following are equivalent.

(a) G is metrizable.

(b) G contains an infinite, convergent sequence (or Cauchy sequence).

(c) G contains a non−discrete, countable subspace (or separable subspace).

(d) G contains a non−discrete, compact subspace (or k−subspace).

(e) G contains a non−discrete subspace S such that some non−isolated point in S is a G_δ−set in S.

Corollary: For an ordered additive group G, the following are equivalent.

(a) G is separable and metrizable.

(b) G is separable.

(c) G is a Lindelöf k−space.

(d) $G-\{x\}$ is Lindelöf for some point $x \in G$.

(*Real number fields*): As is well−known, the definitions of the real number fields are given by several manners, but any real number field is isomorphic to the field $\mathbf{R}$. (Each member of $\mathbf{R} - Q$ is called an *irrational number*).

For the definitions of the real number field $\mathbf{R}$, for example let us recall the following definition (i), (ii), or (iii) (see [9; pp. 21−22]).

(i) $\mathbf{R}$ is defined as the ordered field satisfying the axiom of continuity.

(ii) $\mathbf{R}$ is axiomatically defined as the set satisfying the usual four arithmetic operations and order relation, and the axiom of continuity.

(iii) $\mathbf{R}$ is defined as the set of cuts $(A|B)$ in the rational number field Q as follows (here, if *min* B exists, move it into A):

For real numbers $\alpha = (A_1|A_2)$ and $\beta = (B_1|B_2)$, define

(a) $\alpha < \beta$ iff $A_1 \subset B_1$, but $A_1 \neq B_1$.

(b) $\alpha + \beta = (C_1|C_2)$, where $C_1 = \{x+y : x \in A_1, y \in B_1\}$ and $C_2 = Q - C_1$.

(c) For $\alpha, \beta > 0$, $\alpha \times \beta = (C_1|C_2)$, where $C_2 = \{xy : x \in A_2, y \in B_2\}$ and $C_1 = Q - C_2$. Also, define $\alpha \times (-\beta) = -(\alpha \times \beta)$, $(-\alpha) \times \beta = -(\alpha \times \beta)$, $(-\alpha) \times (-\beta) = \alpha \times \beta$, and $\gamma \times 0 = 0 \times \gamma = 0$ for any γ.

The $(C_1|C_2)$ in (b) and (c) are actually cuts in Q by property of the rationals. We note that Q is dense in $\mathbf{R}$, hence $\mathbf{R}$ is Archimedean, so $\mathbf{R} - Q$ is dense in $\mathbf{R}$, and that $\mathbf{R}$ is actually Dedekind−complete.

Let us consider $\mathbf{R}$ as an ordered field without assuming the axiom of continuity or Archimedes' axiom. Then $\mathbf{R}$ is Archimedean iff Q is dense in $\mathbf{R}$. Also, if $\mathbf{R}$ is Archimedean, then it is second countable, and $\mathbf{R} - Q$ is dense in $\mathbf{R}$ (in view of Theorem 1).

In view of Examples 1 and 2, the author has a question whether $\mathbf{R}$ is Archimedean if (a) $\mathbf{R} - Q$ is dense in $\mathbf{R}$, or (b) $\mathbf{R}$ is second countable (or first countable). The question for case (a) is posed in [9].

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順 序 体 と 距 離 化 可 能 性

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要　　旨

順序体は，線形順序および（その順序による）順序位相をもつ「体」である。順序体は，アルキメデスの公理や連続性公理の観点から，実数体Rの理論において重要な役割を演じてきた。体Rの構造に関して，順序体は，「解析学」や「代数学」などにおいて，代数的および位相的な基盤を与えている。

本論文では，順序体における距離化可能定理や，非アルキメデス的順序体の例を与える。さらに，順序体の周辺として，順序（加法）群における距離化可能性や，実数体の定義を考察する。

キーワード: 順序体，実数体，アルキメデスの公理，連続性公理，距離化可能性，距離空間，リンデレーフ空間