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Partially ordered additive groups and convex sets II

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Abstract

In [8], we consider convexity of subsets in the product groups, or ideals in the product extension rings with respect to the partial order induced by a positive set $T = D_0, D_1, D_2, L_0$, or L. In this paper, we consider convexity of ideals in the product extension rings mainly, focusing D_1 and D_2. We give characterizations for ideals in the product extension rings to be convex for D_1 or D_2. Also, for T, we give a characterization for convexity of principal ideals in the product extensions of the ring of integers.

Keywords: additive group, product extension ring, partial order, positive set, semi-cone, convex set, convex ideal, embedding

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1. Introduction

In this paper, the symbol G means a non-zero additive group (i.e, commutative group under addition) with the identity element 0, and let us call G a *group* briefly. Also, the symbol $G \times G$ means the direct product group.

The symbol R means a non-zero commutative ring with the identity element 1.

For $A, B \subset G$, let $-A = \{-x \mid x \in A\}$, $A + B = \{x + y \mid x \in A, y \in B\}$, and for $A, B \subset R$, let $AB = \{xy \mid x \in A, y \in B\}$. Also, denote $\{0\}$ by 0, and $\{x\}B$ by xB.

The symbol $\mathbb{Z}$, $\mathbb{Z}^*$, $\mathbb{N}$ means the set of integers, non-negative integers, positive integers, respectively.

We recall that a partial order in G (which makes it into a partially ordered group) is determined by a *positive subset* P ([8]); that is, $P + P \subset P$ and $P \cap -P = 0$. Namely, for a positive subset P of G, we induce a partial order $\leq_P$ in G, defining $x \leq_P y$ by $y - x \in P$. Conversely, for a partial order $\leq$ in G, $P = \{x \in G \mid 0 \leq x\}$ is a positive subset of G with $\leq = \leq_P$. For a positive subset P of G with $G = P \cup -P$, $\leq_P$ is a total order in G (that is, G is an *ordered group*).

A positive subset P of R is a *semi-cone* (resp. *cone*) ([5]) if P satisfies $PP \subset P$ (resp. $PP \subset P$ with $R = P \cup -P$). Similarly, a partial order in R is determined by a *semi-cone* P. For a cone P of R, $\leq_P$ is a total order in R with $1 \in P$ (that is, R is an *ordered ring*).

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Let us recall that the group $\mathbb{Z}$ is partially ordered iff it has a positive subset $P \subset \mathbb{Z}^*$ or $P \subset -\mathbb{Z}^*$; equivalently, (*) $P = a_1\mathbb{Z}^* + \cdots + a_m\mathbb{Z}^*$ for some $a_1, \ldots, a_m \in \mathbb{Z}$ with all $a_i \in \mathbb{Z}^*$ or all $a_i \in -\mathbb{Z}^*$ ([9]). Specially, $\mathbb{Z}$ is an ordered group iff it has a positive subset $P = \mathbb{Z}^*$ or $P = -\mathbb{Z}^*$. Also, the ring $\mathbb{Z}$ is partially ordered iff it has a positive subset (or semi-cone) $P \subset \mathbb{Z}^*$; equivalently (*) holds, but all $a_i \in \mathbb{Z}^*$ ([4]). Specially, $\mathbb{Z}$ is an ordered ring iff it is the usual ordered ring.

The symbol P in G means a positive subset. For P of G, G means a partially ordered group with the order $\leq_P$ induced by P, denoted by $\leq$ briefly, unless otherwise stated. For P of G with $P \neq 0$, define $P_0 = \{x \in P \mid x \neq 0\}$.

For a semi-cone P in R, we use the symbol S ([5, 8], etc.).

For $P \neq 0$ of G, let us recall the following subsets of $G \times G$ ([8]). (For $G = R$ and $P = S$, these subsets are considered in [7]).

$D_0 = \{(x, y) \in G \times G \mid x = y \in P\}$,
$D_1 = \{(x, y) \in G \times G \mid 0 \leq y \leq x\}$,
$D_2 = \{(x, y) \in G \times G \mid 0 \leq x \leq y\}$,
$L_0 = P \times P$,
$L = L_0 \cup (P_0 \times G)$.

Obviously, $D_0 = D_1 \cap D_2$, and $D_1 \cup D_2 \subset L_0 \subset L$. Also, D_0, D_1, D_2, L_0, and L are positive subsets of $G \times G$.

For $a, b \in R$, the group $R \times R$ is a ring by the following multiplication ([6]):

$$(x_1, y_1) * (x_2, y_2) = (x_1x_2 + ay_1y_2, x_1y_2 + y_1x_2 + by_1y_2).$$

This ring is called the *product extension ring*, denoted by $(R \ltimes R; a, b)$; in particular, $(R \ltimes R; 0, 0)$ is denoted by $R \ltimes R$. While, we use the symbol $R \otimes R$ for the (usual) *direct product ring* $R \times R$. For these, see [6]. (For characterizations for semi-cones of $\mathbb{Z} \otimes \mathbb{Z}$ or $\mathbb{Z} \ltimes \mathbb{Z}$, see [9]).

We note that the ring $R \otimes R$ or $(R \ltimes R; a, b)$ is the group $G \times G$, putting $G = R$.

Let $P(\neq 0)$ be a positive subset of G. For a subset X of G with $X \ni 0$, X is *convex for* P (or P-*convex*) if whenever $w \leq x \leq y$ and $w, y \in X$ implies $x \in X$ ([8]), here we can assume $w = 0$ for X being a subgroup of G. For a proper subgroup H of G, H is convex for P iff the residue class group G/H has a positive subset $\varphi(P)$ by the natural map φ. In particular, let S be a semi-cone of R. Then, for a proper ideal I of R, I is convex for S iff the residue class ring R/I has a semi-cone $\varphi(S)$ (cf. [2]).

In [8], for a positive subset $T = D_0$, D_1, D_2, L_0, or L of $G \times G$, we study convexity of subsets of $G \times G$, or ideals of the ring $(R \ltimes R; a, b)$. As a continuation of [8], we give further consideration to convexity of ideals of $(R \ltimes R; a, b)$ mainly, and semi-convexity of subsets of $G \times G$, focusing on D_1 and D_2. Also, we give a characterization for T-convexity of principal ideals in $(\mathbb{Z} \ltimes \mathbb{Z}; a, b)$.

2. Embeddings of $\mathbb{Z}$ into partially ordered groups and rings

Let P be a positive subset of G. Let G' be a group, and let g: $G' \to G$ be a group homomorphism. For a positive subset P' of G', g is *order-preserving* for P' if $g(P') \subset P$ (clearly, g is order-preserving for $P' = 0$). We shall say that the group $\mathbb{Z}$ is *semi-embeddable* (resp. *embeddable*) in G (as a partially ordered subgroup) if there exist a group monomorphism g: $\mathbb{Z} \to G$ and a positive subset P' of $\mathbb{Z}$ such that g is order-preserving for P' (resp. $g(P') = g(\mathbb{Z}) \cap P$). If $P' \neq 0$, let us call the group $\mathbb{Z}$ *non-trivially semi-embeddable* in G. In a ring R with a semi-cone S, for the *ring* $\mathbb{Z}$, let us use the similar terminologies, but g is a ring monomorphism and P' is a semi-cone (cf. [4]).

Lemma 2.1. *Let* g: $G' \to G$ *be a group homomorphism. For a positive subset* P of G, let $P' = g^{-1}(P)$ *(possibly* $P' = 0$*). Then*

P' is a positive subset (thus g is order-preserving for P') iff g is injective. For g: $R' \to R$ being a ring homomorphism, if $P = S$ in R, then the similar holds with $P' = S'$ a semi-cone.

Proof. Let us see the latter part. Since g is a ring homomorphism, $S' + S' \subset S'$ and $S'S' \subset S'$. While, $S' \cap -S' = g^{-1}(S \cap -S) = g^{-1}(0)$. Thus $S' \cap -S' = 0$ iff g is an injection. Also, $g(S') \subset S$. Hence, the result holds. □

Let j: $\mathbb{Z} \to R$ be the map defined by $j(n) = n1_R$. As is well-known, j is the only one ring homomorphism from the ring $\mathbb{Z}$ to R. Let $S \neq 0$ in R. For the ring $\mathbb{Z}$, and for $s \in S_0$, define maps

$$f_s : \mathbb{Z} \to R \text{ by } f_s(n) = ns,$$
$$h_s : R \to R \text{ by } h_s(x) = sx$$

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\ j\ } & R \\ {\scriptstyle f_s}\downarrow & & \downarrow{\scriptstyle h_s} \\ R & = & R \end{array}$$

Then $f_s = h_s \circ j$. The maps f_s, h_s are group homomorphisms, and f_s is a ring homomorphism iff $s = 1_R \in S$, and the same holds for h_s. Also, f_s is obviously an injection, and so is j (by $f_s = h_s \circ j$). h_s is an injection if R is an integral domain. Obviously, f_s is order-preserving for $\mathbb{Z}^*$, and so is h_s for S.

The following is a modification of [4, Proposition 2.14].

Proposition 2.2. *The following hold.*

(1) *Let G be a group with a positive subset P. For the group $\mathbb{Z}$, the following are equivalent.*

(a) *$\mathbb{Z}$ is non-trivially semi-embeddable in G.*

(b) *$P \neq 0$.*

(c) *$\mathbb{Z}$ is embeddable in G as an ordered subgroup.*

(2) *Let R be a ring with a semi-cone $S \neq 0$. Then R contains $\mathbb{Z}$ as a subring (via j). For the ring $\mathbb{Z}$, the following are equivalent.*

(a) *$\mathbb{Z}$ is non-trivially semi-embeddable in R.*

(b) *$\mathbb{Z}$ has a semi-cone $j^{-1}(S) \neq 0$.*

(c) *$n1_R \in S_0$ for some $n \in \mathbb{N}$.*

(d) *$j(\mathbb{Z}^*) \cap S_0 \neq \phi$.*

Proof. For (1), (a) ⇒ (b), and (c) ⇒ (a) are obvious. To see (b) ⇒ (c), take $p \in P_0$, then a group monomorphism f_p: $\mathbb{Z} \to G$ is order-preserving for the positive subset $\mathbb{Z}^*$, thus $f_p(\mathbb{Z}^*) = f_p(\mathbb{Z}) \cap P$. Hence (c) holds. For (2), note that the map j: $\mathbb{Z} \to R$ is the only one ring homomorphism, and it is injective (by $S \neq 0$). For the equivalence, (a) ⇔ (b) holds by Lemma 2.1, and so does (b) ⇔ (c) ⇔ (d). □

Corollary 2.3. *Let R be a ring with a semi-cone $S \neq 0$. Then $\mathbb{Z}$ is a subring of R which is an ordered subgroup in R. Also, $\mathbb{Z}$ is an ordered subring of R with the cone $\mathbb{Z}^* \subset S$ iff $1 \in S$.*

Remark 2.4. (1) The ring homomorphism j: $\mathbb{Z} \to R$ need not be injective (by the infinite field), considering $(\mathbb{Z} / p\mathbb{Z})(x)$ (p is a prime number) in [4, Remark 2.16].

(2) In Corollary 2.3, $\mathbb{Z}^* \subset S$ (i.e., $j(\mathbb{Z}^*) \subset S$) need not hold if we omit $1 \in S$. Indeed, let $R = \mathbb{Z}[x]$ be the polynomial ring over the ring $\mathbb{Z}$ in one variable x, and let $S = \{a_i x^i + a_{i+1}x^{i+1} + \cdots + a_n x^n \in R \mid i, n \geq 1, a_i > 0\} \cup \{0\}$. Then S is a semi-cone in R, but $j(\mathbb{Z}^*) \cap S_0 = \phi$ (while, $f_s(\mathbb{Z}^*) \subset S$ for $s \in S_0$).

3. Convexity of sets or ideals

Let G be a group with a positive subset $P \neq 0$. Recall that a subset X of G with $X \ni 0$ is *P-semi-convex* (resp. *P-convex*) if whenever $0 \leq x \leq y \in X$ (resp. $w \leq x \leq y$ and $w, y \in X$) implies $x \in X$ ([8]). For $X \subset P$ or a subgroup X of G, X is P-convex iff it is P-semi-convex. Also, for positive subsets P, P' with $P' \subset P$, if X is P-semi-convex, then X is P'-semi-convex.

For a subset of $G \times G$, similarly we define T-semi-convex sets or T-convex sets for a positive subset T ($\neq 0$) of $G \times G$.

Let p_1, p_2: $G \times G \to G$ be the projections defined by $p_1(x, y) = x$, $p_2(x, y) = y$.

In [8], we give characterizations of semi-convexity of subsets (resp. additive subsets) of $G \times G$ for positive subsets D_0, L (resp. D_1, D_2, L_0). For positive subsets D_1 and D_2, we have the following (cf. [8, Proposition 2.7]).

Proposition 3.1. *Let X be a subset of $G \times G$ with $X \ni (0, 0)$. Then the following hold.*

(1) *If X is D_1-semi-convex, then (i) $p_1(X \cap (G \times 0))$ is P-semi-convex, and (ii) $p_2(X \cap D_1)$ is P-convex with $p_2(X \cap D_1) = p_2(X \cap D_0)$. Conversely, if X is a subgroup of $G \times G$ satisfying (i) and (ii), then X is D_1-convex.*

(2) *If X is D_2-semi-convex, then (i) $p_2(X \cap (G \times 0))$ is P-semi-convex, and (ii) $p_1(X \cap D_2)$ is P-convex with $p_1(X \cap D_2) = p_1(X \cap D_0)$. Conversely, if X is a subgroup of $G \times G$ satisfying (i) and (ii), then X is D_2-convex.*

Proof. (1) For (i), let $0 \leq x \leq y \in p_1(X \cap (G \times 0)$. Then $(0, 0) \leq (x, 0) \leq (y, 0) \in X \cap (G \times 0)$. Since X is D_1-semi-convex, $(x, 0) \in X$, proving (i). For (ii), $p_2(X \cap D_1)$ is P-convex by [8, Proposition 2.2(2)]. By $D_0 \subset D_1$, $p_2(X \cap D_0) \subset p_2(X \cap D_1)$. Let $y \in p_2(X \cap D_1)$. Then $(y, y) \in X$ by [8, Lemma 2.1(2)], and hence $y \in p_2(X \cap D_0)$. Thus $p_2(X \cap D_1) = p_2(X \cap D_0)$. Conversely, assume X is a subgroup of $G \times G$ satisfying (i) and (ii). Let $(0, 0) \leq (x, y) \leq (z, w) \in X$. Since $w \in p_2(X \cap D_1)$, $w \in p_2(X \cap D_0)$ by (ii). Thus $(w, w) \in X$. Since X is a subgroup, $(z - w, 0) = (z, w) - (w, w) \in X$, and hence $z - w \in p_1(X \cap (G \times 0))$. Noting $0 \leq x - y \leq z - w$, $x - y \in p_1(X \cap (G \times 0))$ by (i). Thus $(x - y, 0) \in X$. Noting $0 \leq y \leq w \in p_2(X \cap D_1)$, the convexity of $p_2(X \cap D_1)$ by (ii) implies $y \in p_2(X \cap D_1)$. Hence $y \in p_2(X \cap D_0)$. Thus $(y, y) \in X$. Hence $(x, y) = (x - y, 0)+(y, y) \in X$. Therefore, X is D_1-convex. (2) is similarly shown as (1). □

Proposition 3.2. *Let X be a subset of $G \times G$. For each $i = 1, 2$, assume $X \cap D_0 = X \cap D_i$, then X is D_0-semi-convex iff X is D_i-semi-convex. In particular, for $D_0 = X \cap D_i$, X is D_0-semi-convex, and D_i-semi-convex for each $i = 1, 2$.*

Proof. The if part holds by $D_0 \subset D_1$. For the only if part, to see X is D_1-semi-convex, let $(0, 0) \leq (x, y) \leq (z, w) \in X \cap D_1$. Then $y \leq x$. Also, $z = w$ by the assumption, thus $x \leq y$ by $w - y \leq z - x$. Hence $x = y$. Since X is D_0-semi-convex, $(x, y) \in X$. Hence I is D_1-semi-convex. The result for D_2 is similarly shown. □

Let us consider convexity of ideals of the product extension ring $(R \ltimes R;\, a, b)$.

We assume that R has a non-zero semi-cone S, namely, R has a partial order $\leq$ ($= \leq_s$) induced by $S \neq 0$, unless otherwise stated. Also, we assume the symbol $\mathbb{Z}$ means the ring of integers.

The following proposition is due to [7], which will be often used.

Proposition 3.3. *The following hold.*

(1) *In $R \otimes R$, D_0, D_1, D_2, and L_0 are semi-cones, but L is not a semi-cone.*

(2) *In $(R \ltimes R;\, a, b)$, the following hold.*

 (a) *D_0 is a semi-cone iff $(a + 1)SS \subset S$ and $(a - b - 1)SS = 0$.*

 (b) *D_1 is a semi-cone iff $(b + 2)SS \subset S$ and $(a - b - 1)SS \subset S$.*

(c) *D_2 is a semi-cone iff $aSS \subset S$ and $(b-a)SS \subset S$.*

(d) *L_0 is a semi-cone iff $aSS \subset S$ and $bSS \subset S$.*

(e) *L is a semi-cone iff $aS = bSS = 0$, $S_0S_0 + aR \subset S_0$, and $(S_0 + bR)S \subset S$.*

Remark 3.4. (1) In $(R \ltimes R; a, b)$, we have the following (by Proposition 3.3(2)).

(i) For $SS = 0$, D_0, D_1, D_2, and L_0 are semi-cones, but L is never a semi-cone.

(ii) D_0 and D_2 (or, D_1 and D_2) are simultaneously semi-cones iff $SS = 0$. While, it doesn´t occur that D_0 and L (or, D_1 and L) are simultaneously semi-cones.

(2) For a semi-cone (or positive subset) S in R, there exist a semi-cone S' in $R' = R \ltimes R$ with $S'S' = 0$, and a semi-cone S'' in $R' \otimes R'$ or $(R' \ltimes R'; a, b)$ with $S''S'' = 0$ (by a semi-cone $S' = 0 \times S$, and a semi-cone $S'' = 0 \times S'$ ([7] or [8])).

(3) For R being an integral domain, L is a semi-cone in $(R \ltimes R; a, b)$ iff $a = b = 0$. But, L need not be a semi-cone even if $a = b = 0$ and $S \ni 1$ (see [8, Remark 3.2(2)]).

We can assume R contains the subring $\mathbb{Z}$ (by Proposition 2.2 (2)). In the following lemma, the assumption (i) or (ii) is essential, referring to Remark 3.4 (2).

Lemma 3.5. *For $a \in R$, the following are equivalent if (i) $1 \in S$, or (ii) $a \in \mathbb{Z}$ $(\subset R)$ under $SS \neq 0$, but replace S by $\mathbb{Z}^*$ in (1).*

(1) *$a \in S$ (resp. $a = 0$).*

(2) *$aS \subset S$ (resp. $aS = 0$).*

(3) *$aSS \subset S$ (resp. $aSS = 0$).*

Proof. For (i), the equivalence is obvious. For (ii), (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is obvious. To see (3) $\Rightarrow$ (1), let $aSS \subset S$. Suppose $a \notin \mathbb{Z}^*$. Then $-aSS \subset -S$ and $-aSS \subset S$. Thus, $-aSS = 0$ by $S \cap -S = 0$. Take $t, t' \in S$ with $s = tt' \in S_0$. For an injection $f_s: \mathbb{Z} \to R$ (in the diagram), $f_s(a) = as = 0$, thus $a = 0$, a contradiction. □

The following holds by Proposition 3.3 with Lemma 3.5.

Proposition 3.6. *Let $a, b \in R$. Then (1) ~ (5) below hold in $(R \ltimes R; a, b)$ if (i) $1 \in S$, or (ii) $a, b \in \mathbb{Z}$ under $SS \neq 0$, but replace the order $\leq$ by $\leq_{\mathbb{Z}^*}$.*

(1) *D_0 is a semi-cone iff $0 \leq a + 1$ and $a - b - 1 = 0$ ($\Leftrightarrow (1, 1) * (1, 1) \in D_0$).*

(2) *D_1 is a semi-cone iff $0 \leq b + 2$ and $0 \leq a - b - 1$ ($\Leftrightarrow (1, 1) * (1, 1) \in D_1$).*

(3) *D_2 is a semi-cone iff $0 \leq a$ and $0 \leq b - a$ ($\Leftrightarrow (0, 1) * (0, 1) \in D_2$).*

(4) *L_0 is a semi-cone iff $0 \leq a$ and $0 \leq b$ ($\Leftrightarrow (0, 1) * (0, 1) \in L_0$).*

(5) *L is a semi-cone in $(R \ltimes R; a, b)$ iff $a = b = 0$ ($\Leftrightarrow (0, 1) * (0, 1) = 0$), and $S_0S_0 \subset S_0$.*

Let $(R, \leq)$ be a partially ordered ring, and let I be an ideal of R (possibly, $I = 0$ or R). The ideal I is *convex* if whenever $w \leq x \leq y$ with $w, y \in I$, then $x \in I$ (here, we can assume $w = 0$); in other words, I is *T-convex* with respect to the partial order $\leq$ $(= \leq_T)$ induced by a semi-cone T in R. (For convex ideals in partially ordered rings, see [2, 4, 5, 7, 8], etc.)

Assumption: We assume that $T = D_0$, D_1, D_2, L_0, or L in $(R \ltimes R; a, b)$ is a semi-cone whenever we consider T-convexity of ideals in $(R \ltimes R; a, b)$. (We will use Proposition 3.3 without notice).

Let I be an ideal of $(R \ltimes R; a, b)$. For L, we obtain a characterization for the convexity of I ([7, Theorem 3.5]), and for D_0, D_1, D_2, or L_0, we give certain characterizations in [8]. For D_1 or D_2, we have the following characterization for the convexity of I by Proposition 3.1.

Thmeorem 3.7. *Let I be an ideal of* $(R \ltimes R;\, a, b)$. *Then the following hold.*

(1) *I is D_1-convex iff (i) $p_1(I \cap (R \times 0))$ is S-convex, and (ii) $p_2(I \cap D_1)$ is S-convex with $p_2(I \cap D_1) = p_2(I \cap D_0)$.*

(2) *I is D_2-convex iff (i) $p_2(I \cap (0 \times R))$ is S-convex, and (ii) $p_1(I \cap D_2)$ is S-convex with $p_1(I \cap D_2) = p_1(I \cap D_0)$.*

Remark 3.8. In (1) and (2) of Theorem 3.7, (ii) need not imply (i) in view of (the proof of) [8, Example 3.10(1)]. Also, (i) need not imply (ii). Indeed, let $I = (4, 2) * (\mathbb{Z} \ltimes \mathbb{Z};\, a, b)$ $(=\{(4x + 2ay, 2x + 4y + 2by) \mid x, y \in \mathbb{Z}\})$ and $S = 4\mathbb{Z}^*$ (in [8, Example 3.16]). For (1), let $a = 0$, $b = -2$ (hence, D_1 is a semi-cone). Then $I = \{(4x, 2x) \mid x \in \mathbb{Z}\}$, $p_1(I \cap (\mathbb{Z} \times 0)) = 0$, $p_2(I \cap D_1) = 4\mathbb{Z}^*$ and $p_2(I \cap D_0) = 0$. Thus $p_1(I \cap (\mathbb{Z} \times 0))$ and $p_2(I \cap D_1)$ are S-convex, but $p_2(I \cap D_1) \neq p_2(I \cap D_0)$. For (2), let $a = b = 0$ (hence, D_2 is a semi-cone). Then $I = \{(4x, 2x + 4y) \mid x, y \in \mathbb{Z}\}$. Hence $I \cap (0 \times \mathbb{Z}) = 0 \times 4\mathbb{Z}$, $p_2(I \cap (0 \times \mathbb{Z})) = 4\mathbb{Z} \supset S$. Thus $p_2(I \cap (0 \times \mathbb{Z}))$ is S-convex. But, $p_1(I \cap D_2) = p_1(I \cap D_0) = 8\mathbb{Z}^*$ is not S-convex.

Lemma 3.9. *For an ideal I of $(R \ltimes R;\, a, b)$, $L_0 \subset I$ implies $S \times 0 \subset I$ and $0 \times S \subset I$; conversely, $L_0 \subset I$ if (i) $S \times 0 \subset I$ or (ii) $0 \times S \subset I$ and a is a unit in R.*

Proof. For the latter part, $L_0 = S \times 0 + 0 \times S \subset I$, noting that $(S \times 0) * (0, 1) = 0 \times S$ for (i), and $(0, 1)$ is a unit in $(R \ltimes R;\, a, b)$ for (ii).

For an ideal I of $(\mathbb{Z} \ltimes \mathbb{Z};\, a, b)$, define $I' = I \cap (\mathbb{Z} \times 0)$, and $I'' = I \cap (0 \times \mathbb{Z})$.

Lemma 3.10. *For an ideal I of $(\mathbb{Z} \ltimes \mathbb{Z};\, a, b)$, the following hold.*

(1) *If $p_1(I')$ is non-zero S-convex, then $L_0 \subset I$ (thus $D_1 \subset I$, and I is D_1-convex).*

(2) *If $p_2(I'')$ is non-zero S-convex and a is a unit in $\mathbb{Z}$ (i.e., $a = \pm 1$), then $L_0 \subset I$ (thus $D_2 \subset I$, and I is D_2-convex for $a = 1$).*

Proof. For (1), $S \subset p_1(I')$ by [4, Proposition 3.4]. Thus $S \times 0 \subset I$. Hence $L_0 \subset I$ by Lemma 3.9. (2) is similarly shown, noting $a = 1$ under D_2 being a semi-cone.

The following holds by Theorem 3.7 with Lemma 3.10.

Proposition 3.11. *For an ideal I of $(\mathbb{Z} \ltimes \mathbb{Z};\, a, b)$, the following are equivalent, but for the parenthetic parts, assume $a = 1$.*

(1) *$p_1(I')$ (resp. $p_2(I'')$) is non-zero S-convex.*

(2) *$D_1 \subset I$ (resp. $D_2 \subset I$).*

(3) *$L_0 \subset I$ (resp. $L_0 \subset I$).*

(4) *I is D_1-convex with $I' \neq 0$ (resp. D_2-convex with $I'' \neq 0$).*

(5) *I is L_0-convex with $I' \neq 0$ (resp. L_0-convex with $I'' \neq 0$).*

Remark 3.12. In Proposition 3.11, we have the following.

(1) $p_1(I') \neq 0$ (or $p_2(I'') \neq 0$) is essential (considering a non-zero ideal $I = 0 \times J$ of $(\mathbb{Z} \ltimes \mathbb{Z};\, a, b)$ with $a = 0$, or $I = 0$). Also, $I' \neq 0$ (or $I'' \neq 0$) is essential, reviewing (2) or (3) in Theorem 4.5 later.

For the parenthetic parts, (1) need not imply (2), (3), (4), or (5) without $a = 1$, in view of the example for (2) in Remark 3.8.

(2) (i) D_1 and L_0 (resp. D_2 and L_0) are simultaneously semi-cones in $(\mathbb{Z} \ltimes \mathbb{Z};\, a, b)$ under $0 \leq b \leq a - 1$ (resp. $0 \leq a = 1 \leq b$) by Proposition 3.6, here $\leq \,=\, \leq_{\mathbb{Z}^*}$.

(ii) The parenthetic parts in (1), (2), (3) are equivalent under $a = \pm 1$ (as sets L_0, D_2).

For an ideal I and a semi-cone T in $(R \ltimes R;\, a, b)$, let us recall the following conditions ([8]).

(p_1): $0 \leq x \leq y \in p_1(I \cap T) \Rightarrow (x, 0) \in I$.

(p_2): $0 \le x \le y \in p_2(I \cap T) \Rightarrow (0, x) \in I$.

The following is shown in [8, Proposition 3.13] (with the proof of (2) there).

Lemma 3.13. *Let I be an ideal of $(R \ltimes R;\, a, b)$. For D_1, (p_1) implies (p_2), and the convexity of I. For D_2, (p_1) and (p_2) imply the convexity of I.*

We note that (p_1) (or (p_2)) need not imply the convexity of I for D_2 ([8, Remark 3.17(1)]). Also, note that the convexity of I for D_0, D_1 or D_2 need not imply (p_1) or (p_2); see Example 3.19 (3) later.

The following is due to [8, Lemma 3.11] with Lemma 3.13.

Lemma 3.14. *For D_1 (resp. D_2), suppose that there exists $(p, q) \in (R \ltimes R;\, a, b)$ satisfying condition (d_1): $(x, 0) \le (x, y) + (y, y) * (p, q)$ for any $(x, y) \in D_1$ (resp. (d_2): $(0, y) \le (x, y) + (x, x) * (p, q)$ for any $(x, y) \in D_2$). Then, for an ideal I of $(R \ltimes R;\, a, b)$, I is convex for D_1 (resp. D_2) iff (p_1) and (p_2) hold. Here, for D_1, (p_2) can be deleted.*

Remark 3.15. Let I be an ideal of $R \otimes R$ (thus, $I = p_1(I) \times p_2(I)$, and D_1 and D_2 semi-cones). Then, for each $i = 1, 2$, I is D_i-convex iff (p_1) and (p_2) hold in $R \otimes R$ ([8, Proposition 2.10 or 3.3]) by means of Lemma 3.14 (with the similar proof of [8, Lemma 3.11]), noting (d_1) (resp. (d_2)) holds in $R \otimes R$ for $(p, q) = (1, -1)$ (resp. $(p, q) = (-1, 1)$).

For each $i = 1, 2$, let $D_i(a, b)$ be the set of elements $(p, q) \in (R \ltimes R;\, a, b)$ satisfying (d_i) in Lemma 3.14.

For $(p, q) \in (R \ltimes R;\, a, b)$, let us consider the following conditions.

$D_1(a, b; p, q)$: $(p + (b + 1)\, q + 1)\, S \subset S$ and $((a - b - 1)\, q - 1)\, S \subset S$.
$D_2(a, b; p, q)$: $(p + aq + 1)\, S \subset S$ and $((b - a + 1)\, q - 1)\, S \subset S$.

Remark 3.16. It doesn´t occur that $D_1(a, b; p, q)$ and $D_2(a, b; p', q)$ simultaneously hold by $S \neq 0$. Also, neither $D_1(a, b; p, q)$ nor $D_2(a, b; p', q)$ holds for $(a - b - 1)\, q = 0$ (by $S \neq 0$).

The following lemma is routinely shown.

Lemma 3.17. *For each $i = 1, 2$, $(p, q) \in D_i(a, b)$ iff $D_i(a, b; p, q)$ holds. In particular, if $(a - b - 1)\, S = 0$, $D_i(a, b) = \phi$ (but D_2 is not a semi-cone if $SS \neq 0$).*

Proposition 3.18. *Let I be an ideal of $(R \ltimes R;\, a, b)$. Assume $a - b - 1$ is a unit in R. Then, for each $i = 1, 2$, $D_i(a, b)$ is infinite (thus $D_i(a, b) \neq \phi$), and moreover, I is D_1-convex (resp. D_2-convex) iff (p_1) (resp. (p_1) and (p_2)) holds.*

Proof. Let c be the inverse of $a - b - 1$ in R. For D_1, put $q = c\,(s + 1)$ for $s \in S$, and $p = -(1 + b)\, q + s'$ for $s' \in S$. Then $(p, q) \in D_1(a, b)$ by Lemma 3.17, and $D_1(a, b)$ is an infinite set. The latter half follows from Lemma 3.14. For D_2, noting $b - a + 1$ is a unit, the result for D_2 is similarly shown. □

Example 3.19. Related to a result that for each $i = 1, 2$, under $D_i(a, b) \neq \phi$, an ideal I of $(R \ltimes R;\, a, b)$ is D_i-convex iff (p_1) and (p_2) hold (that is, Lemma 3.14), we have the following.

(1) For each $i = 1, 2$, though $D_i(a, b) \neq \phi$, there exists an ideal I in $(R \ltimes R;\, a, b)$ such that neither (p_1) nor (p_2) holds (also I is neither D_1-convex nor D_2-convex).

(2) For each $i = 1, 2$, though $D_i(a, b) = \phi$, there exists a D_i-convex ideal I in $(R \ltimes R; a, b)$ such that (p_1) and (p_2) hold.

(3) For each $i = 1, 2$, though $D_i(a, b) = \phi$, there exists a D_i-convex (or D_0-convex) ideal I in $(R \ltimes R; a, b)$ such that neither (p_1) nor (p_2) holds (even if $p_i(I)$ and $p_i(I \cap D_j)$ are convex for $i, j = 1, 2$).

Indeed, let $R = \mathbb{Z}$. To see (1), for $a = 0$, $b = -2$ (resp. $a = b = 1$), D_1 (resp. D_2) is a semi-cone. Since $D_1(0, -2; 1, 1)$ (resp. $D_2(1, 1; 0, 1)$) holds, $D_i(a, b) \neq \phi$ $(i = 1, 2)$ by Lemma 3.17. Let $I = (4, 2) * (\mathbb{Z} \ltimes \mathbb{Z}; a, b)$ $(= \{(4x + 2ay, 2x + 4y + 2by) \mid x, y \in \mathbb{Z}\})$ and $S = 4\mathbb{Z}^*$ (in Remark 3.8). Then I doesn´t satisfy (p_1) or (p_2) (noting $(0, 0) \leq (4, 4) \leq (8, 4) \in I \cap D_1$ and $(0, 0) \leq (4, 4) \leq (4, 12) \in I \cap D_2$, but $(4, 4), (4, 0), (0, 4) \notin I$). As other example, let $J = (2, 2) * (\mathbb{Z} \ltimes \mathbb{Z}; a, b)$ and $S = \mathbb{Z}^*$. Then J is never T-convex, and never satisfy (p_1) or (p_2) for $T = D_0, D_1, D_2, L_0$ or L (noting for $a, b \in \mathbb{Z}$, $(0, 0) \leq (1, 1) \leq (2, 2) \in J \cap D_0$, but $(1, 1), (1, 0), (0, 1) \notin J$).

For (2) and (3), suppose $a - b - 1 = 0$. Then $D_i(a, b) = \phi$ for $i = 1, 2$ (by Lemma 3.17).

To see (2), consider the ideal I and $S = 4\mathbb{Z}^*$ in (1). Then $I = \{(4x + 2ay, 2x + 2ay + 2y) \mid x, y \in \mathbb{Z}\}$ by $a - b - 1 = 0$. Also, D_1 is a semi-cone in $(\mathbb{Z} \ltimes \mathbb{Z}; a, b)$ under $a + 1 \in S$, so let $a = -1$, $b = -2$. Then (p_1) and (p_2) hold (by $(4, 0), (0, 4) \in I$), and thus I is D_1-convex by Lemma 3.13. On the other hand, D_2 is not a semi-cone (by $a - b - 1 = 0$ and $SS \neq 0$). So, let us give a way to make D_2 be a semi-cone in some $(R' \ltimes R'; a', b')$ with $a' - b' - 1' = 0$, but R' has a semi-cone S' with $S'S' = 0$. Let $R' = \mathbb{Z} \ltimes \mathbb{Z}$, and let $S' = 0 \times 4\mathbb{Z}^*$. For $n \in \mathbb{Z}$, let $n' = (n, 0) \in R'$. For $a' = -1'$, $b' = -2'$, $a' - b' - 1' = 0$. Since $S'S' = 0$, D_2 (or D_1) is a semi-cone in $(R' \ltimes R'; a', b')$ with respect to S'. Let $I' = (4', 2') * (R' \ltimes R'; a', b')$. Then, for I', (p_1) and (p_2) hold (by $((0, 4), (0, 0)), ((0, 0), (0, 4)) \in I'$). Thus I' is D_2-convex as well as D_1-convex in $(R' \ltimes R'; a', b')$ with respect to the semi-cones D_1, D_2 by S'. Hence, for D_2, I' with the semi-cone S' is a desired ideal.

To see (3), let $I = (1, 1) * (\mathbb{Z} \ltimes \mathbb{Z}; a, b)$ and $S = \mathbb{Z}^*$. Then $I = \{(x, x) \mid x \in \mathbb{Z}\}$ by $a - b - 1 = 0$, and $D_0 = I \cap D_i$ $(i = 1, 2)$. Since $a - b - 1 = 0$, D_0, D_1 (resp. D_2) are semi-cones if $a + 1 \in S$ (resp. $SS = 0$). Then I is D_i-convex $(i = 0, 1, 2)$ by Proposition 3.2, and $p_i(I) = \mathbb{Z}$ and $p_i(I \cap D_j)$ $(= S)$ $(i, j = 1, 2)$ are S-convex, but for $a = 1$, $b = 0$, I satisfies neither (p_1) nor (p_2). Here, for D_2, let $R' = \mathbb{Z} \ltimes \mathbb{Z}$, and $S' = 0 \times \mathbb{Z}^*$, and $1' = (1, 0)$. Then $I' = (1', 1') * (R' \ltimes R'; 1', 0)$ with the semi-cone S' is a desired ideal.

The following holds by Lemmas 3.14 and 3.17.

Proposition 3.20. *Let I be an ideal of $(R \ltimes R; a, b)$. For some $p, q \in (R \ltimes R; a, b)$, assume $D_1(a, b; p, q)$ (resp. $D_2(a, b; p, q)$) holds. Then I is D_1-convex (resp. D_2-convex) iff (p_1) (resp. (p_1) and (p_2)) holds.*

The following proposition is due to Proposition 3.13 in [8], but (2) and (3) are modifications (indeed, for (2) (resp. (3)), consider $D_1(a, b; p, q)$ (resp. $D_2(a, b; p, q)$) in Proposition 3.20, putting $(p, q) = (1, -1)$ for (i), and $(p, q) = (1, 1)$ for (ii)).

Proposition 3.21. *In $(R \ltimes R; a, b)$, the following hold.*

(1) *For D_0, if (p_1) holds, then I is convex. Conversely, the convexity of I implies (p_1) and (p_2) hold if $a - b - 1$ is a unit and $SS = 0$.*

(2) *For D_1, if (p_1) holds, then I is convex. Conversely, the convexity of I implies (p_1) and (p_2) hold if (i) $(1 - b)\, S \subset S$ and $(b - a)\, S \subset S$ under $SS = 0$, or (ii) $(a - b - 2)\, S \subset S$ and $(b + 3)\, S \subset S$, but assume $SS = 0$ for $(b + 2)\, SS \not\subset S$. Here, D_1 is a semi-cone under (i) or (ii).*

(3) *For D_2, if (p_1) and (p_2) hold, then I is convex. Conversely, the convexity of I implies (p_1) and (p_2) hold if (i) $(2 - a)\, S \subset S$ and $(a - b - 2)\, S \subset S$ under $SS = 0$, or (ii) $(a + 2)\, S \subset S$ and $(b - a)\, S \subset S$, but assume $SS = 0$ for $aSS \not\subset S$. Here, D_2 is a semi-cone under (i) or (ii).*

(4) *For L_0, I is convex iff (p_1) and (p_2) hold.*

Remark 3.22. For Proposition 3.21, in (i) or (ii) in (2) (resp. (3)), $SS = 0$ is essential for D_1 (resp. D_2) to be a semi-cone (by Proposition 3.6 with Lemma 3.5). Also, (i) or (ii) of (2), (3) need not imply (p_1) or (p_2). Indeed, for the ideal I in the proof of Example 3.19(1), take $a = b = 1$ (resp. $a = 1$, $b = -3$) in (i) of (2) (resp. (3)), and take $a = 0$, $b = -2$, or $a = 3$, $b = -3$ (resp. $a = b = 1$, or $a = b = -1$) in (ii) of (2) (resp. (3)). For the ideal J in the proof, take a, b satisfying (i) or (ii) of (2), (3). Here, for the ideal I

or J, in case of (i), or (ii) under $SS = 0$, use the similar way in the proof of Example 3.19(2). Hence, (i) or (ii) of (2), (3) need not imply (p_1) or (p_2) as in view of the proof of Example 3.19(1).

A partially ordered ring $(R, \leq)$ is *Archimedean* if for any two elements $x > 0$ and $y > 0$ in R, there exists $n \in \mathbb{N}$ such that $y < nx$ (see [1], [2], etc.).

Proposition 3.23. *Let* $(R, \leq)$ *be a partially ordered ring which is Archimedean. Let I be an ideal of* $(R \ltimes R;\ a, b)$. *Then the following hold. (In (1), for D_2, see Remark 3.24(1) below).*

(1) *Assume* $1 \in S$ *and* $a - b - 1 \neq 0$. *Then* $D_1(a, b)$ *is an infinite set* (*thus* $D_1(a, b) \neq \phi$), *and I is D_1-convex iff* (p_1) *holds.*

(2) *Assume* $I \cap D_1 \neq I \cap D_0$ (*resp.* $I \cap D_2 \neq I \cap D_0$). *Then I is D_1-convex* (*resp. D_2-convex*) *iff* (p_1) (*resp.* (p_1) *and* (p_2)) *holds.*

Proof. (1) Since D_1 is a semi-cone, $(a - b - 1)\, SS \subset S$, thus $a - b - 1 > 0$ by $1 \in S$. Since $(R, \leq)$ is Archimedean, for $a - b - 1 > 0$ and $1 > 0$, there exists $n \in \mathbb{N}$ with $1 < n(a - b - 1)$. Put $q = n + s$ for $s \in S$, and $p = -(1 + b)q + s' - 1$ for $s' \in S$. Then $(p, q) \in D_1(a, b)$ by Lemma 3.17, and $D_1(a, b)$ is an infinite set. The latter half follows from Lemma 3.14.

(2) The if part holds by Lemma 3.13, so we see the only if part. For D_1, assume I is convex. To see (p_1), let $0 < x \leq y \in p_1(I \cap D_1)$. Then $y = p_1(y, y')$ for some $(y, y') \in I \cap D_1$.

(i) Case $y \neq y'$. Then $x > 0$ and $y - y' > 0$. Since $(R, \leq)$ is Archimedean, there exists $n \in \mathbb{N}$ such that $n(y - y') > x$. Then $(0, 0) < (x, 0) \leq n(y, y') \in I$. Thus $(x, 0) \in I$ by the D_1-convexity of I.

(ii) Case $y = y'$. Since $I \cap D_1 \neq I \cap D_0$, there exists $(z, w) \in I \cap D_1$ with $z \neq w$. Then $(y + z, y' + w) \in I \cap D_1$ with $y + z \neq y' + w$, and $0 < x < y + z \in p_1(I \cap D_1)$. Thus $(x, 0) \in I$ by (i).

For D_2, assume I is D_2-convex. Then (p_2) holds by the same way as the above proof of (p_1) for D_1. To see (p_1), let $0 \leq x \leq y \in p_1(I \cap D_2)$. Then $y = p_1(y, y')$ for some $(y, y') \in I \cap D_2$. Then, $0 \leq y \leq y'$ and $0 \leq x \leq y' \in p_2(I \cap D_2)$, thus $(0, x) \in I$ by (p_2). While, $(0, 0) \leq (x, x) \leq (y, y') \in I$, then $(x, x) \in I$ by the D_2-convexity of I. Thus $(x, 0) = (x, x) - (0, x) \in I$. Hence ($p_1$) holds. □

Remark 3.24. (1) Related to Proposition 3.23(1), the following modification holds in view of the proof there, without $(R, \leq)$ being Archimedean.

For D_1 (resp. D_2), if $1 \leq (a - b - 1)r$ (resp. $1 \leq (b - a + 1)r$) for some $r \in R$, then $D_1(a, b)$ (resp. $D_2(a, b)$) is infinite, and moreover, an ideal I of $(R \ltimes R;\ a, b)$ is D_1-convex (resp. D_2-convex) iff (p_1) (resp. (p_1) and (p_2)) holds.

In particular, assume $0 \leq a - b - 2$ for D_1, and $1 \in S$ (or $0 \leq b - a$) for D_2. Then $D_1(a, b)$ is infinite, and so is $D_2(a, b)$, putting $r = 1$ (note $0 \leq b - a$ for D_2 being a semi-cone with $1 \in S$). Thus, an ideal I of $(R \ltimes R;\ a, b)$ is D_1-convex (resp. D_2-convex) iff (p_1) (resp. (p_1) and (p_2)) holds (cf. [8]). But, this need not hold for D_1 or D_2 under $a - b - 1 = 0$ by Example 3.19(3).

(2) The assumption $I \cap D_i \neq I \cap D_0$ ($i = 1, 2$) in Proposition 3.23(2) is essential by Example 3.19(3), noting the rings $\mathbb{Z}$ and R there are Archimedean.

(3) For an ideal I of $(R \ltimes R;\ a, b)$, assume $I \cap D_0 = I \cap D_1$ (resp. $I \cap D_0 = I \cap D_2$). Then I is D_0-convex iff I is D_1-convex (resp. D_2-convex) by Proposition 3.2, here $SS = 0$ for the parenthetic part (by Remark 3.4(1)(ii)).

4. Convexity of principal ideals in $(\mathbb{Z} \ltimes \mathbb{Z};\ a, b)$

Let S be a non-zero semi-cone in $\mathbb{Z}$. We will consider convexity of an ideal $I = (p, q) * (\mathbb{Z} \ltimes \mathbb{Z};\ a, b)$ generated by a single element (p, q) ($\neq (0, 0)$) for $T = D_0, D_1, D_2, L_0$, or L. We assume that T is a semi-cone in $(\mathbb{Z} \ltimes \mathbb{Z};\ a, b)$ for T-convexity of I, and let us recall the following fact (Proposition 3.6), here $\leq\, = \,\leq_{\mathbb{Z}^*}$.

Fact. D_0 is a semi-cone iff $0 \leq a + 1$ and $a - b - 1 = 0$,
D_1 is a semi-cone iff $0 \leq b + 2$ and $0 \leq a - b - 1$,
D_2 is a semi-cone iff $0 \leq a$ and $0 \leq b - a$,
L_0 is a semi-cone iff $0 \leq a$ and $0 \leq b$, and

L is a semi-cone iff $a = b = 0$.

Obviously, $I = (p, q) * (\mathbb{Z} \ltimes \mathbb{Z}; a, b) = \{(px + aqy, qx + (p + bq)y) \mid x, y \in \mathbb{Z}\}$, here we can assume $p \geq 0$ (by $I = (-p, -q) * (\mathbb{Z} \ltimes \mathbb{Z}; a, b)$).

For $(c_1, c_2) \in \mathbb{Z} \times \mathbb{Z}$, $(c_1, c_2) \in I$ iff the simultaneous linear equation

(4.1) $\quad px + aqy = c_1, \quad qx + (p + bq)y = c_2$

is solvable in $\mathbb{Z}$. Let d be the determinant of the coefficients of (4.1), that is,

$$d = p(p + bq) - aq^2 = p^2 + bqp - aq^2.$$

Hereafter, $I = (p, q) * (\mathbb{Z} \ltimes \mathbb{Z}; a, b)$ with $(p, q) \neq (0, 0)$, and $d = p^2 + bqp - aq^2$. Also, S means a non-zero semi-cone of $\mathbb{Z}$.

Lemma 4.1. *Suppose $d = p^2 + bqp - aq^2 \neq 0$. Then, for $T = D_0, D_1, D_2$ or L_0, I is T-convex iff $T \subset I$.*

Proof. The if part is obvious. So we will show the only if part. Noting $d \neq 0$, for $(c_1, c_2) \in \mathbb{Z} \times \mathbb{Z}$, $(c_1, c_2) \in I$ iff

(4.2) $\quad x = (c_1(p + bq) - aqc_2)/d \in \mathbb{Z}, \quad y = (pc_2 - qc_1)/d \in \mathbb{Z}.$

For any $s \in S$, $m, n \in \mathbb{Z}^*$, $(m|d|s, n|d|s) \in I$, where $|d|$ is the absolute value of d (indeed, for $(c_1, c_2) = (m|d|s, n|d|s)$, x and y in (4.2) belong to $\mathbb{Z}$ by $|d|/d \in \mathbb{Z}$, hence $(m|d|s, n|d|s) \in I$). Also, notice $D_0 \subset D_i \subset L_0$ $(i = 1, 2)$.

(i) For $T = D_0$,

(4.3) $\quad (0, 0) \leq (s, s) \leq (|d|s, |d|s) \in I$

holds. Since I is D_0-convex, $(s, s) \in I$ for any $s \in S$ by (4.3). Hence $D_0 \subset I$.

(ii) For $T = D_1$, (4.3) and

(4.4) $\quad (0, 0) \leq (s, 0) \leq (2|d|s, |d|s) \in I$

hold. Since I is D_1-convex, $(s, s), (s, 0) \in I$ for any $s \in S$. Thus $D_1 \subset I$.

(iii) For $T = D_2$, (4.3) and

(4.5) $\quad (0, 0) \leq (0, s) \leq (|d|s, 2|d|s) \in I$

hold. Since I is D_2-convex, $(s, s), (0, s) \in I$ for any $s \in S$. Thus $D_2 \subset I$.

(iv) For $T = L_0$, (4.3), (4.4) and (4.5) hold. Since I is L_0-convex, for any $s \in S$, $(s, s), (s, 0), (0, s) \in I$. Thus $L_0 \subset I$. □

The following holds by Proposition 3.11 with Lemma 4.1.

Corollary 4.2. *Suppose $d \neq 0$. Then I is D_1-convex iff I is L_0-convex. Also, under $a = 1$, I is D_2-convex iff I is L_0-convex.*

Now, let us recall that S can be expressed as $S = \sum_{i=1}^{r} a_i\mathbb{Z}^*$ for some $a_1, \ldots, a_r \in \mathbb{N}$ ([4, Proposition 2.9]). Define $d_S = \gcd(a_1, \ldots, a_r) \in \mathbb{N}$.

Noting $d_S\mathbb{Z} = \sum_{i=1}^{r} a_i\mathbb{Z}$ is the smallest ideal of $\mathbb{Z}$ containing S, d_S is independent from choice of its expression.

We note that a non-zero ideal J of $\mathbb{Z}$ is S-convex iff $J \supset S$ ($\Leftrightarrow J \supset d_S\mathbb{Z}$) ([4, Proposition 3.4]).

For $x, y \in \mathbb{Z}$, the symbol $x \mid y$ means $y = xr$ for some $r \in \mathbb{Z}$.

Lemma 4.3. *For $m, n \in \mathbb{Z}$ with $m \neq 0$, $m \mid ns$ for all $s \in S$ iff $m \mid nd_S$.*

Proof. Obviously, $m \mid ns$ for all $s \in S \Longleftrightarrow m\mathbb{Z} \supset ns\mathbb{Z}$ for all $s \in S \Longleftrightarrow m\mathbb{Z} \supset n(\sum_{i=1}^{r} a_i\mathbb{Z}) = nd_S\mathbb{Z} \Longleftrightarrow m \mid nd_S$. □

Lemma 4.4. *Suppose $d = p^2 + bqp - aq^2 \neq 0$. Then*

(4.6) $\quad D_0 \subset I \iff d \mid d_S \gcd(p + (b - a)q, p - q)$

(4.7) $\quad D_1 \subset I \iff d \mid d_S \gcd(p, q)$

(4.8) $\quad D_2 \subset I \iff d \mid d_S \gcd(p, q)$

(4.9) $\quad L_0 \subset I \iff d \mid d_S \gcd(p, q)$

Proof. Recalling $(c_1, c_2) \in I$ iff $x, y \in \mathbb{Z}$ in (4.2), for $c \in \mathbb{Z}$,

$$\begin{aligned}
(c, c) \in I &\iff c(p + bq - aq)/d \in \mathbb{Z},\ c(p - q)/d \in \mathbb{Z} \\
&\iff d \mid c\gcd(p + (b - a)q, p - q) \\
(c, 0) \in I &\iff c(p + bq)/d \in \mathbb{Z},\ c(-q)/d \in \mathbb{Z} \\
&\iff d \mid c\gcd(p, q) \\
(0, c) \in I &\iff (-aqc)/d \in \mathbb{Z},\ pc/d \in \mathbb{Z} \\
&\iff d \mid c\gcd(p, aq)
\end{aligned}$$

We note

(4.10) $\quad \gcd(\gcd(p, aq), \gcd(p + (b - a)q, p - q)) = \gcd(p, q),$

indeed, the left hand of (4.10) is $\gcd(p, aq, p + (b - a)q, p - q) = \gcd(p, q)$.

For D_0, $D_0 \subset I \Leftrightarrow (s, s) \in I$ for all $s \in S \Leftrightarrow d \mid s\gcd(p + (b - a)q, p - q)$ for all $s \in S \Leftrightarrow d \mid d_S\gcd(p + (b - a)q, p - q)$ (by Lemma 4.3).

For D_1, D_2 or L_0, noting $(0, s) = (s, 0) * (0, 1)$, we have $D_1 \subset I$; $D_2 \subset I$; $L_0 \subset I$ iff for all $s \in S$, $(s, 0) \in I$; (s, s), $(0, s) \in I$; $(s, 0) \in I$, respectively. Thus, similarly $D_1 \subset I$ iff $d \mid d_S\gcd(p, q)$; $D_2 \subset I \Leftrightarrow d \mid d_S\gcd(\gcd(p, aq), \gcd(p + (b - a)q, p - q)) \Leftrightarrow d \mid d_S\gcd(p, q)$ (by (4.10)); $L_0 \subset I$ iff $d \mid d_S\gcd(p, q)$. □

In [8, Remark 3.18], we observe the L_0-convexity of $I = (p, q) * (\mathbb{Z} \ltimes \mathbb{Z}; a, b)$. For $T = D_0, D_1, D_2$, or L_0, we give the following characterization for the T-convexity of I.

Thmeorem 4.5. *The following hold, here* $\le = \le_{\mathbb{Z}^*}$ *in (2) and (3).*

(1) *Case* $d \neq 0$.
   - (a) *I is* D_0*-convex iff* $d \mid d_S(p - q)$ ($\Leftrightarrow D_0 \subset I$)
   - (b) *I is* D_1*-convex iff* $d \mid d_S\gcd(p, q)$ ($\Leftrightarrow D_1 \subset I$).
   - (c) *I is* D_2*-convex iff* $d \mid d_S\gcd(p, q)$ ($\Leftrightarrow D_2 \subset I$).
   - (d) *I is* L_0*-convex iff* $d \mid d_S\gcd(p, q)$ ($\Leftrightarrow L_0 \subset I$).

(2) *Case* $d = 0$ *with* $p \neq 0$ (*thus*, $q \neq 0$). *Assume* $p > 0$.
   - (a) *I is* D_0*-convex iff (i)* $p \neq q$ ($\Leftrightarrow I \cap D_0 = \{(0, 0)\}$) *or (ii)* $p = q$ *and* $p \mid d_S$ ($\Leftrightarrow D_0 \subset I$).
   - (b) *I is* D_1*-convex iff (i)* $0 < p < q$ ($\Leftrightarrow I \cap D_1 = \{(0, 0)\}$), *(ii)* $p = q$ and $p \mid d_S$ ($\Leftrightarrow I \cap D_1 = D_0$), *or (iii)* $q < 0$ ($\Leftrightarrow I \cap D_1 = \{(0, 0)\}$).
   - (c) *I is* D_2*-convex iff* $q < p$ ($\Leftrightarrow I \cap D_2 = \{(0, 0)\}$).
   - (d) *I is* L_0*-convex iff* $q < 0$ ($\Leftrightarrow I \cap L_0 = \{(0, 0)\}$).

(3) *Case* $d = 0$ *with* $p = 0$ (*thus* $a = 0$). *Assume* $q > 0$. *For* $i = 0, 1$, *I is* D_i*-convex with* $I \cap D_i = \{(0, 0)\}$. *Also, for* $T = D_2$ *or* L_0, *I is T-convex iff* $q \mid d_S$ ($\Leftrightarrow 0 \times S \subset I$).

Proof. (1) follows from Lemmas 4.1 and 4.4, here $D_0 \subset I$ iff $d \mid d_S(p - q)$ for D_0 being a semi-cone, because $b - a + 1 = 0$ by *Fact*.

(2) Evidently, the equation (4.1) is equivalent to $px + aqy = c_1$, $dy = pc_2 - qc_1$. By $d = 0$, $(c_1, c_2) \in I$ implies

(4.11) $\quad pc_2 = qc_1 \quad (q \neq 0)$

(a) Let $p \neq q$. Let $(c, c) \in I \cap D_0$. Then $pc = qc$ by (4.11). Thus $c = 0$, hence $I \cap D_0 = \{(0, 0)\}$. Thus I is D_0-convex.

Let $p = q$ (thus, $a = b + 1$ by $d = 0$). Then $I = \{(px, px) \mid x \in \mathbb{Z}\}$, and $(0, 0) \leq (s, s) \leq (ps, ps) \in I$ for any $s \in S$. Thus, if I is D_0-convex, then $(s, s) \in I$, and hence $D_0 \subset I$. Conversely, if $D_0 \subset I$, then I is D_0-convex. But, $D_0 \subset I \Longleftrightarrow p \mid s$ for all $s \in S$ $\Longleftrightarrow p \mid d_S$ (by Lemma 4.3). Hence, I is D_0-convex iff $p \mid d_S$.

(b) Let $0 < q < p$. Then I is not D_1-convex. Indeed, suppose I is D_1-convex. Then $(0, 0) \leq (s, s) \leq (ps, qs) \in I$. Since I is D_1-convex, $(s, s) \in I$. But, since $ps \neq qs$, $(s, s) \notin I$ (by (4.11)), a contradiction.

Let $p = q$. Then $I = \{(px, px) \mid x \in \mathbb{Z}\}$. Thus, if I is D_1-convex, then $D_0 = I \cap D_1$, and the converse holds by Proposition 3.2. Also, $D_0 \subset I \Leftrightarrow p \mid d_S$ as is seen in the proof of (ii) in (a). Thus (ii) holds.

Let $0 < p < q$. Let $(c_1, c_2) \in I \cap D_1$. Then $0 \leq c_2 \leq c_1$. Since $(c_1, c_2) \in I$, we have $pc_2 = qc_1$ by (4.11). Thus $c_1 = 0$, and hence $c_2 = 0$. Hence $I \cap D_1 = \{(0, 0)\}$. Thus I is D_1-convex.

Let $q < 0$. Then $I \cap D_1 = \{(0, 0)\}$ (indeed, for $(c_1, c_2) \in I \cap D_1$, $c_1 = c_2 = 0$ by (4.11)). Thus I is D_1-convex.

(c) We can assume $p \neq q$ (for $p = q$, D_2 is not a semi-cone by $a - b - 1 = 0$).

Let $0 < p < q$. Then I is not D_2-convex. Indeed, let $0 \neq s \in S$. Then $(ps, qs) \in I \cap D_2$. Further, $(0, 0) \leq (0, s) \leq (ps, qs)$. But $(0, s) \notin I$ by (4.11), a contradiction.

Let $q < p$. Then $I \cap D_2 = \{(0, 0)\}$, noting (4.11) (indeed, let $(c_1, c_2) \in I \cap D_2$. Then $0 \leq c_1 \leq c_2$. But $pc_2 = qc_1$ implies $c_2 = c_1 = 0$). Obviously, if $I \cap D_2 = \{(0, 0)\}$, then I is D_2-convex.

(d) Let $q > 0$. Then I is not L_0-convex. Indeed, let $s \in S_0$. Then $(ps, qs) \in I \cap L_0$, and $(0, 0) \leq (0, s) \leq (ps, qs) \in I$. But $(c_1, c_2) = (s, 0) \notin I$ by (4.11), a contradiction.

Let $q < 0$. Then $I \cap L_0 = \{(0, 0)\}$ (indeed, let $(c_1, c_2) \in I \cap L_0$, then $c_1 \geq 0$, $c_2 \geq 0$. Thus $c_1 = c_2 = 0$ by (4.11)). Then I is L_0-convex.

In view of the above proof, we have the equivalence (including the only if part) in (a), (b), (c), or (d).

(3) Since $p = a = 0$, $I = 0 \times q\mathbb{Z}$. Thus for $i = 0, 1$, $I \cap D_i = \{(0, 0)\}$, hence I is D_i-convex. Also, for $T = D_2$ or L_0, I is T-convex iff $0 \times S \subset I$ ($\Leftrightarrow S \subset q\mathbb{Z} \Leftrightarrow q \mid d_S$), noting $(0, 0) \leq (0, s) \leq (0, qs) \in I$ for all $s \in S$. □

Finally, for L, let us consider convexity of the ideal I in $(\mathbb{Z} \ltimes \mathbb{Z}; a, b)$, here $a = b = 0$ by *Fact*.

Proposition 4.6. *Assume* $p \geq 0$. *Then* I *is* L*-convex iff either (i)* $p = 0$ *and* $q \mid d_S$ ($\Leftrightarrow I = 0 \times q\mathbb{Z}$ *with* $q\mathbb{Z} \supset S$), *or (ii)* $p = 1$ ($\Leftrightarrow$ $I = \mathbb{Z} \ltimes \mathbb{Z}$).

Proof. $I = \{(px, qx + py) \mid x, y \in \mathbb{Z}\}$ (by $a = b = 0$).

Let $p = 0$. Then $I = 0 \times q\mathbb{Z}$ ($\Leftrightarrow p = 0$). Assuming $q > 0$, I is L-convex iff $q\mathbb{Z}$ is S-convex ($\Leftrightarrow q\,\mathbb{Z} \supset S \Leftrightarrow q \mid d_S$).

Let $p = 1$. Then, obviously $I = \{(x, qx + y) \mid x, y \in \mathbb{Z}\} = \mathbb{Z} \ltimes \mathbb{Z}$ ($\Leftrightarrow p = 1$).

Let $p > 1$. Then I is not L-convex. Indeed, suppose I is L-convex. Take $s \in S_0$. Then for all $n \in \mathbb{Z}$,

$$(0, 0) \leq (s, n) \leq (sp, sq) \in I.$$

Since I is L-convex, $(s, n) \in I$ for all n. In particular, $(s, 2), (s, 1) \in I$. Thus $(0, 1) = (s, 2) - (s, 1) \in I$. Hence $(0, 1) = (px, qx + py)$ for some $x, y \in \mathbb{Z}$, which yields $py = 1$, a contradiction. □

Let S' be any non-zero semi-cone in $\mathbb{Z}$. Similarly, define $d_{S'} \in \mathbb{N}$, and subsets D_0', D_1', D_2', L_0', L' of $(\mathbb{Z} \ltimes \mathbb{Z}; a, b)$ by S' instead of S, here assume these subsets are semi-cones. Obviously, $d_S = d_{S'}$ need not imply $S = S'$. But, the following holds in view of Theorem 4.5 and Proposition 4.6, here $(A, A') = (B, B')$ means $A = B$ and $A' = B'$.

Proposition 4.7. *Suppose* $d_S = d_{S'}$. *Then the ideal* I *of* $(\mathbb{Z} \ltimes \mathbb{Z}; a, b)$ *is* T*-convex iff it is* T'*-convex for* $(T, T') = (D_0, D_0')$, (D_1, D_1'), (D_2, D_2'), (L_0, L_0'), *or* (L, L').

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半順序加法群と凸集合 II

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要　旨

論文[1]において，直積群の部分集合や積拡大環のイデアルの凸性について，正集合 $T = D_0, D_1, D_2, L_0$, または L から誘導された半順序の観点から考察した。

本稿では，D_1 や D_2 に焦点を絞り，主に積拡大環のイデアルの凸性を考察をする。D_1 や D_2 に対し，積拡大環のイデアルの凸性について，特徴付けを与える。さらに，T に対して，整数環の積拡大環における単項イデアルの凸性について特徴付けを与える。

キーワード: 加法群，積拡大環，半順序，正集合，半コーン，凸集合，凸イデアル，埋め込み

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