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Products of symmetric spaces

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Abstract

Symmetric spaces are classical and very important spaces among generalized metric spaces. In this paper, for a subspace of a product of symmetric spaces, we give a characterization for the subspace to be symmetric. Also, as applications, we consider necessary and sufficient conditions for products of symmetric spaces to be symmetric.

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Introduction

We assume that spaces are regular T_1, and maps are continuous and onto.

Symmetric spaces are classical and important spaces among generalized metric spaces. For a metric space (X, d), all ϵ-spheres are open by the triangle inequality (i.e., $d(x, z) \leq d(x, y) + d(y, z)$). This fact is used for many topological properties of metric spaces. As a useful generalization of metric spaces, we consider symmetric spaces admitting a distance function which need not satisfy the triangle inequality (hence, every ϵ-sphere is not necessarily open).

The first author showed in [17] that every subset of a countable symmetric space need not be symmetric, and that every product of a countable symmetric space with a countable metric space need not be symmetric. While, he showed in [18] that a space X is locally compact symmetric if and only if $X \times Y$ is symmetric for every symmetric space Y.

In this paper, for a subset A of a product of symmetric spaces, we give a characterization for A to be symmetric. Also, as applications, we consider necessary and sufficient conditions for products of symmetric spaces to be symmetric.

We shall give main definitions, *symmetric spaces* and *weak topologies.*

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A space X is *symmetric* (or *symmetrizable*) [1] (or [13]), if there exists a non-negative real valued function d defined on $X \times X$ satisfying the following (a), (b), and (c).

(a) $d(x,y) = 0$ iff $x = y$;

(b) $d(x,y) = d(y,x)$;

(c) $G \subset X$ is open in X if and only if, for each $x \in G$, there exists $n \in N$ such that $S_n(x) \subset G$, where $S_n(x) = \{y \in X : d(x,y) < 1/n\}$; that is, the topology of the space X is equivalent to $\{G \subset X : \forall x \in G, \exists n \in N \text{ with } S_n(x) \subset G\}$.

Also, X is *semi-metric* (or *semi-metrizable*) [10] if we replace (c) by "For $A \subset X$, $x \in clA$ iff $d(x,A) = 0$".

For the above function d on X, let us say that (X,d) is symmetric (resp. semi-metric), or X is symmetric (resp. semi-metric) with respect to d.

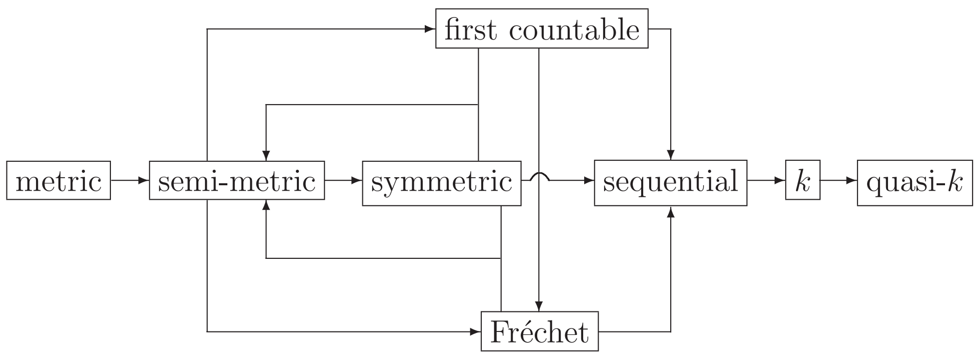
We note that (X,d) is semi-metric if and only if it is symmetric, and for each $x \in X$, $\{int\ S_n(x) : n \in N\}$ is a local base at x in X. Also, a space is semi-metric if and only if it is a hereditarily symmetric space ([17]). Around symmetric spaces, see [5], [13], [21], etc.

For a cover $\mathcal{P}$ of a space X, we recall that X *is determined by* $\mathcal{P}$ [6], if X has the *weak topology* with respect to $\mathcal{P}$ [2]; that is, $G \subset X$ is open in X if $G \cap P$ is open in P for each $P \in \mathcal{P}$. Here, we can replace "open" by "closed". We call such a cover $\mathcal{P}$ a *determining cover* [24, 25] (or [9]). For basic properties of weak topologies, see [20, 23], etc.

A space X is a *sequential space* [4] (resp. *k-space*; *quasi-k-space* [12]) if X has a determining cover by compact metric sets (resp. compact sets; countably compact sets). It is well-known that every sequential space (resp. k-space; quasi-k-space) is precisely a quotient image of a metric space (resp. locally compact space; M-space); see [4] (resp. [2]; [12]). Here, a space is an *M-space* if it admits a quasi-perfect map onto a metric space.

We have the following diagram around symmetric spaces. Here, a space X is *Fréchet-Urysohn* (or *Fréchet*) if for any $A \subset X$ and any $x \in \overline{A}$, there exists a sequence $\{x_n\}$ in A converging to x.

Diagram



We conclude this introduction by giving some examples.

We recall that a space X is *quasi-metric* (or Δ*-metric*) if there exists a function d on X satisfying (a) and (c), and the triangle inequality (instead of the symmetry (b)). In a quasi-metric space (X,d), all ϵ-spheres are open. Thus, every quasi-metric space is first countable. The class of quasi-metric spaces is productive and hereditary. For quasi-metric spaces, see [5], [8], or [13]. Also, we recall that a space X is *developable* if it has a sequence $\{\mathcal{G}_n : n \in N\}$ of open covers of X such that each point $x \in X$ has a local base $\{St(x,\mathcal{G}_n) : n \in N\}$, where $St(x,\mathcal{G}_n) = \cup\{G \in \mathcal{G}_n : x \in G\}$. In this case, the space X is semi-metric with respect to $d(x,y) = inf\{1/n : y \in St(x,\mathcal{G}_n) \text{ for some } n \in N\}$.

Example A. (1) The *Butterfly space* (or *Bow-tie space*) (R^2,d), R is the set of all real numbers, is semi-metric with respect to a function d such that $d(x,y) = |x-y| + a(x,y)$ if either x or y is on the x-axis, otherwise, $d(x,y) = |x-y|$, where $|x-y|$ is the ordinary Euclidean distance and $a(x,y)$ is the radian measure of the smallest non-negative angle formed by a line through x and y with a horizontal line. Then all ϵ-spheres are open, but the space is not metric.

(2) The *Sorgenfrey line* (R, d) is quasi-metric with respect to a function d defined by $d(x, y) = 1$ if $x > y$, and $d(x, y) = y - x$ if $y \geq x$, but the space is not symmetric; see [7], etc.

(3) There exists a developable space (hence, it is semi-metric with respect to a function such that all ϵ-spheres are open), but it is not quasi-metric; see [5; Example 10.4].

(4) There exists a developable quasi-metric space which is not metric, because every meta-compact developable space is quasi-metric; see [5].

We recall the following elementary fact (*); see [1], [5], or [13].

(*) Let $f : X \to Y$ be a quotient map which is compact (resp. finite-to-one). If X is a metric (resp. symmetric) with respect to d, then Y is symmetric with respect to $\rho(y, y') = d(f^{-1}(y), f^{-1}(y'))$. Here, f is *compact* (resp. *finite-to-one*) if $f^{-1}(y)$ is compact (resp. finite) for each $y \in Y$.

Example B. Let T be the topological sum of $\{L_n : n = 0, 1, 2, \cdots\}$, where L_n are the convergent sequence $\{1/n : n \in N\} \cup \{0\}$.

The *Sequential fan* S_ω is the quotient space obtained from T by identifying all the limit points in T to a single point.

The *Arens' space* S_2 is the quotient space obtained from T by identifying $1/n \in L_0$ with each $0 \in L_n$ $(n \geq 1)$.

The space S_2 is symmetric by the fact (*), because S_2 is a quotient finite-to-one image of the metric space T, but S_2 is not Fréchet. While, the space S_ω is a Fréchet space (actually, S_ω is a closed image of T), but it is not symmetric, for S_ω is not first countable.

We note that every subset of a symmetric space need not be symmetric, for a subset $(S_2 - L_0) \cup \{0\}$ of S_2 is not symmetric. Also, note that every perfect image of a symmetric space need not be symmetric, for S_ω is a perfect image S_2/L_0 of S_2.

For spaces which contains a copy of S_2 or S_ω, and their applications, see [20, 23], etc.

Main theorem

In the following lemmas, Lemma 1.2 is a classical result due to [13]. Lemmas 1.1, 1.3, and 1.4 are well-known or easily shown. Lemmas 1.5 and 1.6 are routinely shown, so we shall omit their proofs.

Lemma 1.1. Let (X, d) be a symmetric space. If A is a closed subspace of X, then A is symmetric with respect to the restriction $d|A$ of d on A.

Lemma 1.2. Every countably compact symmetric space is metric.

Lemma 1.3. Let (X, d) be a symmetric space. Then the following are equivalent.
(a) (X, d) is a semi-metric space;
(b) X is first countable;
(c) For any subspace A of X, A is semi-metric with respect to $d|A$.

Lemma 1.4. Let (X_i, d_i) $(i = 1, 2)$ be semi-metric spaces. Let d be the function d on $X_1 \times X_2$ defined by $d((x, y), (x', y')) = \sqrt{d_1(x, x')^2 + d_2(y, y')^2}$. Then $X_1 \times X_2$ is semi-metric with respect to d.

Lemma 1.5. Let d be a function on X satisfying (a) $d(x, y) = 0$ iff $x = y$; (b) $d(x, y) = d(y, x)$; and (c') If $x \in G$ with G open in X, some $S_n(x) \subset G$. Then $A \subset X$ is symmetric with respect to $d|A$ if A has a determining cover $\mathcal{P}$ such that each $P \in \mathcal{P}$ is symmetric with respect to $d|P$.

Lemma 1.6. Let $\mathcal{P}$ be a determining cover of X, and let $\mathcal{Q}$ be a cover of X such that each element of

$\mathcal{P}$ is contained in an element of $\mathcal{Q}$. Then $\mathcal{Q}$ is a determining cover of X.

Theorem 1.7. Let (X_i, d_i) $(i = 1, 2)$ be symmetric spaces. For $A \subset X_1 \times X_2$, the following are equivalent.

(a) A is symmetric;
(b) A is sequential;
(c) A is a quasi-k-space.

Proof. It suffices to show that (c) implies (a). Let d be the function on $X_1 \times X_2$ defined by $d((x, y), (x', y')) = \sqrt{d_1(x, x')^2 + d_2(y, y')^2}$. Let us show that $(A, d|A)$ is symmetric. Since A is a quasi-k-space, A has a determining cover $\mathcal{P}$ by countably compact sets in $X_1 \times X_2$. For each $P \in \mathcal{P}$, $P \subset p_1(P) \times p_2(P)$, where p_i is the projection from $X_1 \times X_2$ onto X_i. Also, $p_i(P)$ is closed in X_i. Indeed, let $q_i \notin p_i(P)$. To show some $S_n(q_i)$ doesn't meet $p_i(P)$, suppose that $S_n(q_i)$ meet $p_i(P)$ for any $n \in N$. Take a point $x_{in} \in S_n(q_i) \cap p_i(P)$ for each $n \in N$, thus the sequence $\{x_{in}\}$ converges to the point q_i. Since $p_i(P)$ is countably compact, the sequence in $p_i(P)$ has an accumulation point y_i in $p_i(P)$, thus $y_i = q_i$, a contradiction. Then, $p_i(P)$ is symmetric with respect to $d_i|p_i(P)$ by Lemma 1.1. Since $p_i(P)$ is first countable by Lemma 1.2, $p_i(P)$ is semi-metric with respect to $d_i|p_i(P)$ by Lemma 1.3. Thus $p_1(P) \times p_2(P)$ is semi-metric with respect to $d_P = d|(p_1(P) \times p_2(P))$ by Lemma 1.4. Hence, $(p_1(P) \times p_2(P)) \cap A$ is semi-metric with respect to $d_P|((p_1(P) \times p_2(P)) \cap A)$ by Lemma 1.3. But, A has a determining cover $\{(p_1(P) \times p_2(P)) \cap A : P \in \mathcal{P}\}$ by Lemma 1.6. Then A is symmetric with respect to $d|A$ by Lemma 1.5.

Corollaries

As applications of Theorem 1.7, we give some corollaries which are shown by means of this theorem.

Corollary 2.1. Let $f_i : X_i \to Y_i$ $(i = 1, 2)$ be quotient compact maps such that X_i are metric. For $A \subset Y_1 \times Y_2$, the following are equivalent.

(a) A is symmetric;
(b) A is a quasi-k-space;
(c) A is a quotient compact image of a metric space.

Proof. Since every quotient compact image of a metric space is symmetric, it suffices to show that (a) implies (c). Since A is a k-space, A has a determining cover $\mathcal{P}$ by compact sets in $X_1 \times X_2$. But, for each $P \in \mathcal{P}$, $P \subset (K \times L)$ for some compact sets K and L. Let $h = f_1 \times f_2$. Since $f_1^{-1}(K)$ and $K \times L$ are k-spaces, $h|h^{-1}(K \times L)$ is quotient (see [3; Theorem 3.3.28]). Thus, for each $P \in \mathcal{P}$, $h|h^{-1}(P)$ is quotient since P is closed in $K \times L$. Then, since A has a determining cover $\mathcal{P}$, $h|h^{-1}(A)$ is quotient. Thus A is a quotient compact image of a metric space $h^{-1}(A)$.

For Corollary 2.2, the result for $X \times Y$ being a k-space is shown in [13] (or [17]) by different methods. For Corollary 2.3, the similar is shown in [17].

Corollary 2.2. Let X and Y be symmetric spaces. Then $X \times Y$ is symmetric if and only if it is a quasi-k-space.

Corollary 2.3. Let X be a symmetric space (in particular, a quotient compact image of a metric space). Then $A \subset X$ is symmetric if and only if A is a quasi-k-space.

Every product of a sequential space with a locally countably compact space is a quasi-k-space ([16]). Thus the following corollary holds.

Corollary 2.4. Let X and Y be symmetric spaces. If Y is locally countably compact, then $X \times Y$ is symmetric.

A cover $\mathcal{P}$ of a space X is a *k-network* [15], if for any compact set K and any open set G such that

$K \subset G$, $K \subset \cup\mathcal{F} \subset G$ for some finite $\mathcal{F} \subset \mathcal{P}$. A space X is an $\aleph$-*space* [15] (resp. $\aleph_0$-*space* [11]) if X has a σ-locally finite k-network (resp. countable k-network). For k-networks, see [20], etc. Every first countable $\aleph$-space is metric ([14]). We recall that every k-and-$\aleph_0$-space is characterized as a quotient image of a separable metric space ([11]).

For k-spaces having certain point-countable k-networks, in [22] the first author gave some necessary and sufficient conditions for their products to be k-spaces. In this paper, let us consider only the following basic and original type: For k-and-$\aleph$-spaces X_1 and X_2, $X_1 \times X_2$ is a k-and-$\aleph$-space if and only if the property (i), (ii), or (iii) in Corollary 2.5 below holds ([19]). Thus the following corollary holds.

Corollary 2.5. Let X_i $(i = 1, 2)$ be symmetric $\aleph$-spaces (in particular, quotient compact images of separable metric spaces). Then $X_1 \times X_2$ is symmetric if and only if the following property (i), (ii), or (iii) holds.

(i) X_1 and X_2 are metric;

(ii) X_1 or X_2 is locally compact metric;

(iii) X_1 and X_2 have a countable determining closed cover by locally compact (metric) sets.

Corollary 2.6. Let (X_i, d_i) $(i = 1, 2)$ be symmetric spaces having σ-locally finite covers $\{S_{n_{x_i}}(x_i) : x_i \in X_i\}$, where n_{x_i} is some integer for each $x_i \in X_i$. Then $X_1 \times X_2$ is symmetric if and only if the property (i), (ii), or (iii) in Corollary 2.5 holds.

Proof. Let K be a compact set in X_i. Then for any $p \in K$, and for any $n \in N$, $p \in int_K(S_n(p) \cap K)$ by Lemmas 1.1, 1.2, and 1.3. Then the covers $\{S_n(x_i) : n \geq n_{x_i}, x_i \in X_i\}$ are σ-locally finite k-networks, thus X_i are $\aleph$-spaces. Hence, the corollary holds by Corollary 2.5.

Corollary 2.7. Let X be a countable symmetric space, and let Y be a metric space. Then $X \times Y$ is symmetric if and only if X is metric, or Y is locally compact.

Proof. Since X is countable, it has a countable k-network $\{S_n(x) : x \in X, n \in N\}$, thus X is an $\aleph_0$-space. While, Y is locally compact if it has a countable determining closed cover $\mathcal{P}$ by locally compact sets. Indeed, since Y is first countable, each point $y \in Y$ has a decreasing countable local base. We can assume that the cover $\mathcal{P}$ is increasing. Thus, it is routinely shown that some nbd of y is contained in an element of $\mathcal{P}$. Hence Y is locally compact. Thus, the corollary holds by Corollary 2.5.

The following remark shows a reason why we don't consider Corollary 2.5 in terms of "subset" of the product (as in Theorem 2.1).

Remark 2.8. There exists a symmetric $\aleph_0$-space X such that for some (non-trivial) subset A of X^2, A is not metric, and A has no countable determining closed cover by locally compact sets (hence, A doesn't satisfy any of the properties in Corollary 2.5). Indeed, let X be the topological sum of the Arens' space S_2 and the space Q of rationals. Then, X is a symmetric $\aleph_0$-space. For an infinite compact set K in X, let $A = X \times K$. Then, X is a symmetric $\aleph_0$-space, and A is symmetric in X^2 by Corollary 2.4, but A is not metric, and X has no countable determining closed cover by locally compact sets, for the space Q is not locally compact.

Remark 2.9. Any finite product of the symmetric space S_2 is symmetric (by Corollary 2.5), but the infinite countable product of S_2 is not symmetric. Indeed, a countable symmetric space X is metric if the countable product X^ω of X is symmetric (because, if X is not compact, X contains a closed copy of N, then X^ω contains a closed copy $X \times N^\omega$. Thus, $X \times N^\omega$ is symmetric, hence X is metric by Corollary 2.7).

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