Title	The Time of Arrival of Electrons in the Double-Slit Experiment(author's version)
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Citation	Physical Review A, 77(1): 014102-1-014102-4
Issue Date	2008-01
URL	http://hdl.handle.net/2309/90487
Publisher	
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The Time of Arrival of Electrons in the Double-Slit Experiment

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Using Nelson's stochastic mechanics, quantum motion of electrons in the double-slit experiment is studied numerically. It is found that not only the distribution of arrival positions but also that of arrival times at the screen forms an interference pattern. Quantum-mechanical presence time and arrival time are also calculated in comparison with stochastic-mechanical results.

PACS numbers: 03.65.†, 03.65.Ta, 03.65.Xp

There is no doubt that position and time are the most fundamental quantities for describing the motion of an object. In classical mechanics, the equation of motion determines the position of an object as a function of time, $\mathbf{x}(t)$. Then it is straightforward to calculate the time of arrival of the object moving from a position $\mathbf{x}_i$ to another $\mathbf{x}_f$.

In quantum mechanics, however, the time of arrival cannot be obtained within the standard procedure of calculating an expectation value because there is no well-established self-adjoint time operator [1, 2] It should be noted that Pauli pointed out that there is no self-adjoint time operator conjugate to a Hamiltonian bounded from below [1, 3].

From an Experimental point of view, it is evident that the time of arrival can be measured, in principle, by means of time of flight technique, etc.. Therefore, a complete theory must be able to predict the time of arrival. In this sense, the lack of established method of obtaining the time of arrival may be regarded as a point of imperfection in quantum mechanics. To overcome this difficulty, serious efforts has been made by many theorists to establish a theory predicting the time of arrival [1, 2, 4–7].

An alternative way of calculating the time of arrival is to employ Nelson's stochastic mechanics [8, 9]. Since stochastic mechanics gives a set of trajectories (sample paths), the time of arrival for each trajectory is obtained in the same way of classical mechanics. A few recent works showed that stochastic mechanics is actually a powerful tool for calculating the tunneling time [10, 11].

In this brief report, we consider the time of arrival for the double-slit experiment. Since the establishment of quantum mechanics the double-slit experiment by electrons has been one of the most famous Gedankenexperiment for demonstrating the fundamental concept of quantum mechanics. The realization of such double-slit experiments has been reported in recent years [12, 13]. The double-slit experiment performed by Tonomura *et al.* recorded arrival positions of electrons one by one and confirmed that the accumulated arrival positions on the screen indeed reproduce the interference pattern [13].

The formation of a double-slit interference pattern has been studied based on the stationary states. However, as shown by Tonomura *et al.*[13], the interference pattern is composed of the arrival positions of individual electrons. In this case the quantum motion of each electron should be described by a time-dependent wave packet rather than a stationary wave. Then not only the position but also the time of arrival of electrons at the screen becomes a fundamental subject. Experimentally, by measuring the time of arrival for each electron, one will obtain a time of arrival distribution. The purpose of our present report is to predict numerically the time of arrival distribution for a double-slit experiment by using stochastic mechanics.

In stochastic mechanics, a possible trajectory of an electron is expressed as a sample path calculated by the Ito stochastic differential equation given by [8, 9]

$$d\mathbf{x}(t) = \mathbf{b}(\mathbf{x}(t), t)dt + d\mathbf{w}(t). \tag{1}$$

$dw_i(t)$ represents the Brownian motion defined by

$$\langle dw_i(t) \rangle = 0, \quad \langle dw_i(t) dw_j(t) \rangle = \frac{\hbar}{m}\delta_{ij}dt, \tag{2}$$

where $\hbar$ is the Planck constant divided by 2π, and m the mass of an electron. The drift velocity, $\mathbf{b}(\mathbf{x}, t)$ is given by

$$\mathbf{b}(\mathbf{x}, t) = \frac{\hbar}{m}\nabla(\Im + \Re)\ln\psi(\mathbf{x}, t), \tag{3}$$

where $\psi(\mathbf{x}, t)$ is the solution of the Schrödinger equation.

Using stochastic mechanics outlined above, we calculate trajectories (sample paths) of electrons for the double-slit experiment. We assume that electrons moving along the y axis pass through a double slit placed at $x = 0$ and then reach a screen placed at $x = X$. For simplicity, we further assume that the wave function passing through the double-slit is approximated by a couple of two-dimensional Gaussian wave packets [14, 15]:

$$\psi(x, y, t) = \psi_1(x, y, t) + \psi_2(x, y, t), \tag{4}$$

where

$$\psi_j(x,y,t)=\frac{1}{2\sqrt{\pi a}(1+i\xi t)}\exp\left\{-\left[\frac{1}{2a}\frac{(x-s_j)^2+y^2}{1+i\xi t}\right]+i\left(\frac{k_0 y+\omega_0 t}{1+i\xi t}\right)\right\},\quad (j=1,2), \tag{5}$$

$s_1=-s, s_2=s$, $(0,\pm s)$ being the coordinates of the center of slits, $\xi=\hbar/(ma)$, and $\omega_0=\hbar k_0^2/(2m)$. Setting $\psi_j=\exp(R_j+iS_j)$, $(j=1,2)$ with

$$\mathbf{u}_j=\frac{\hbar}{m}\nabla R_j,\quad \mathbf{v}_j=\frac{\hbar}{m}\nabla S_j,\quad (j=1,2), \tag{6}$$

we have the osmotic velocity $\mathbf{u}$ and the current velocity $\mathbf{v}$ by [8]

$$\mathbf{u} = \frac{\mathbf{u}_1+\mathbf{u}_2}{2}+\frac{\sinh(R_1-R_2)(\mathbf{u}_1-\mathbf{u}_2)-\sin(S_1-S_2)(\mathbf{v}_1-\mathbf{v}_2)}{2(\cosh(R_1-R_2)+\cos(S_1-S_2))}, \tag{7}$$

$$\mathbf{v} = \frac{\mathbf{v}_1+\mathbf{v}_2}{2}+\frac{\sinh(R_1-R_2)(\mathbf{v}_1-\mathbf{v}_2)+\sin(S_1-S_2)(\mathbf{u}_1-\mathbf{u}_2)}{2(\cosh(R_1-R_2)+\cos(S_1-S_2))}. \tag{8}$$

From the relation $\mathbf{b}=\mathbf{u}+\mathbf{v}$ we obtain the drift velocity as follows:

$$b_x = \frac{\hbar}{m}\left[\frac{\hbar t/m-a}{a^2+(\hbar t/m)^2}\right]x+\frac{\hbar}{m}\left[\frac{s(\hbar/m-a)}{a^2+(\hbar t/m)^2}\right]\frac{\sinh[-2asx/(a^2+(\hbar t/m)^2)]-\sin[2tsx/(a^2+(\hbar t/m)^2)]}{\cosh[-2asx/(a^2+(\hbar t/m)^2)]-\cos[2tsx/(a^2+(\hbar t/m)^2)]}, \tag{9}$$

$$b_y = \frac{\hbar}{m}\left[\frac{\hbar t/m-a}{a^2+(\hbar t/m)^2}\right]y+\frac{p_0}{m}\left[\frac{a(a+\hbar t/m)}{a^2+(\hbar t/m)^2}\right]. \tag{10}$$

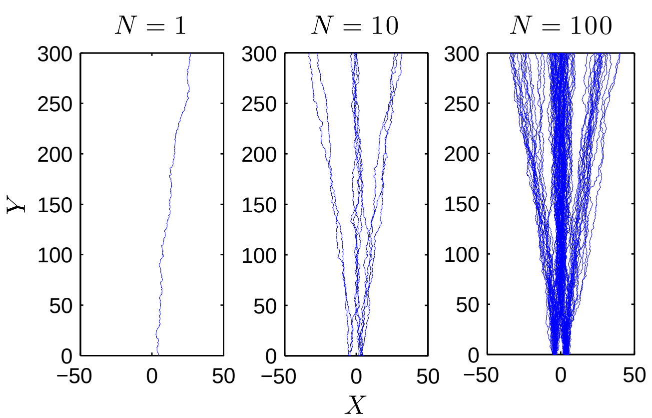

FIG. 1: Sample paths in double-slit experiment. N denotes the number of sample paths.

Using Eqs. (4)-(10) we have solved the stochastic differential equation (1) numerically. Examples of calculated sample paths are shown in Fig. 1. From the figure one can see that the interference pattern becomes clearer as the number of sample paths increases.

Now, we consider the time of arrival. Before showing our numerical results, let us introduce two expressions that have been used for discussion of the time of arrival [1].

One is the *presence time* defined by

$$\langle T\rangle^{\mathrm{p}}_{X,Y}=\int_0^\infty T\rho^{\mathrm{p}}_{X,Y}(T)dT \tag{11}$$

with the presence time distribution

$$\rho^{\mathrm{p}}_{X,Y}(T)dT=\frac{p(X,Y,T)dT}{\int_0^\infty p(X,Y,T)dT}, \tag{12}$$

where $p(X,Y,T)=|\psi(X,Y,T)|^2$. $\rho^{\mathrm{p}}_{X,Y}(T)dT$ is proportional to the probability density of finding an electron at a detector position (X,Y) at a time interval $T\sim T+dT$. Although the presence time seems to be a simple definition of a time of arrival in quantum mechanics, Eq. (12) has not been derived within the standard procedure of calculating an observable due to the lack of a general time operator.

The other one is the *arrival time* defined by

$$\langle T\rangle^{\mathrm{a}}_{X,Y}=\int_0^\infty T\rho^{\mathrm{a}}_{X,Y}(T)dT, \tag{13}$$

with the arrival time distribution

$$\rho^{\mathrm{a}}_{X,Y}(T)dT=\frac{|\mathbf{j}(X,Y,T)\cdot d\mathbf{S}|dT}{\int_0^\infty|\mathbf{j}(X,Y,T)\cdot d\mathbf{S}|dT}, \tag{14}$$

where $\mathbf{j}(X,Y,T)$ represents the current density and $d\mathbf{S}$ is the surface element of a detector. The arrival time formula is derived from Bohmian mechanics [1, 16].

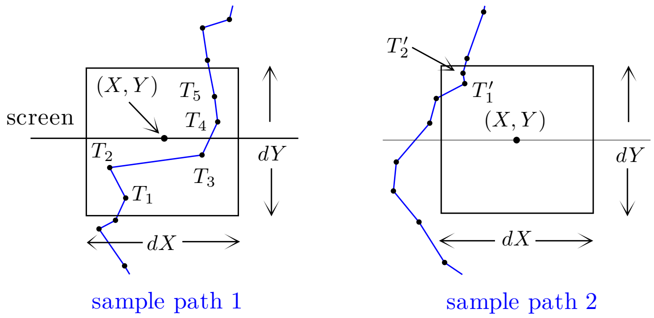

FIG. 2: The first counting scheme and the multiple counting scheme. Detector's width and area for the point (X, Y) are dX and $dXdY$, respectively. The traversing time in the first counting scheme is $(T_3 + T_4)/2$ for sample path 1 while the existing times in the multiple counting scheme are $(T_1 + T_2 + T_3 + T_4 + T_5)/5$ for the sample path 1 and $(T'_1 + T'_2)/2$ for the sample path 2.

In contrast to quantum mechanics, stochastic mechanics has no fundamental difficulty in defining a time of arrival because stochastic mechanics gives particle trajectories or sample paths. It should be noted, though, that there are two schemes for defining the detection of a particle at a fixed point (see Fig. 2). One is the "first counting scheme". This scheme assumes that a particle is counted by a detector when its sample path traverses detector's surface for the first time. Therefore, the time of arrival in the first counting scheme for the sample path 1 in Fig. 2 is given by $(T_3 + T_4)/2$, where $T_1, \cdots, T_5$ represent the arrival time at each point. Defining $N^{\rm f}_{X,Y}(T)dTdX$ as the number of sample paths traversing the detector placed at $y = Y$ with the acceptance width $X - dX/2 \sim X + dX/2$ in the time interval $T \sim T + dT$, we introduce the "first counting time distribution":

$$\rho^{\rm f}_{X,Y}dT = \frac{N^{\rm f}_{X,Y}(T)dT}{\int_0^\infty N^{\rm f}_{X,Y}(T)dT}. \tag{15}$$

Using $\rho^{\rm f}_{X,Y}$ the time of arrival in the first counting scheme is given by

$$\langle T\rangle^{\rm f}_{X,Y} = \int_0^\infty T\rho^{\rm f}_{X,Y}(T)dT. \tag{16}$$

The other one is the "multiple counting scheme". This scheme assumes that a particle is counted by a detector in a probabilistic manner when a sample path comes in the detector. Therefore, the time of arrival in the multiple counting scheme for the sample path 1 in Fig. 2 is given by $(T_1 + T_2 + T_3 + T_4 + T_5)/5$. It should be noted that the sample path 2 contributes to the time of arrival in the multiple counting scheme but does not so in the first counting scheme. Defining $N^{\rm m}_{X,Y}(T)dXdYdT$ as the number of the sample paths coming in the detector area $(X - dX/2, Y - dY/2) \sim (X + dX/2, Y + dY/2)$ at time $T \sim T + dT$ we introduce the "multiple counting arrival time distribution" :

$$\rho^{\rm m}_{X,Y}dT = \frac{N^{\rm m}_{X,Y}(T)dT}{\int_0^\infty N^{\rm m}_{X,Y}(T)dT}. \tag{17}$$

It should be noted that the number of $\int N^{\rm m}_{X,Y}(T)dT$ is greater than the total number of the sample paths into the detector. The multiple counting time is represented by

$$\langle T\rangle^{\rm m}_{X,Y} = \int_0^\infty T\rho^{\rm m}_{X,Y}(T)dT. \tag{18}$$

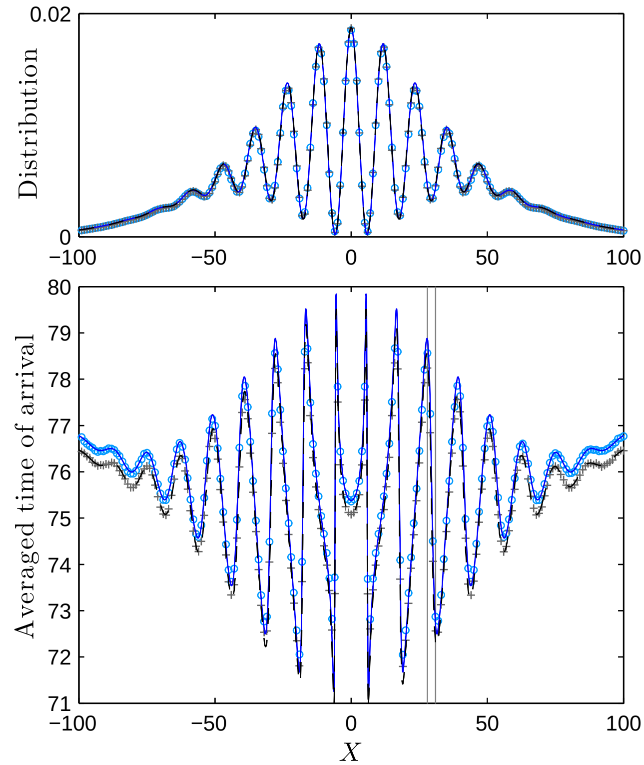

FIG. 3: The number of electrons arrived at the screen (upper figure) and the time of arrival (lower figure). The multiple counting time (∘), first counting time (+), presence time (solid line), and arrival time (dotted line) are plotted.

In Fig. 3, we show typical numerical results. Parameter values have been chosen as $p_0 = 8$, $Y = 600$, $a = 2$, and $s = 20$ with $\hbar = m = 1$. The total number of sample paths is $N = 1.0 \times 10^8$. The upper figure shows the distribution of electrons arrived at the screen, i.e. the interference pattern in the double-slit experiment. Though we have calculated four different distributions $\rho^\alpha_{X,Y}$ (α = p,a,f,m), their values agree with each other almost perfectly.

The lower figure represents the time of arrival of electrons at the screen. It is found that the first counting time $\langle T\rangle^{\rm f}_{X,Y}$ coincides with the arrival time $\langle T\rangle^{\rm a}_{X,Y}$ calculated by the probability current density in the quantum

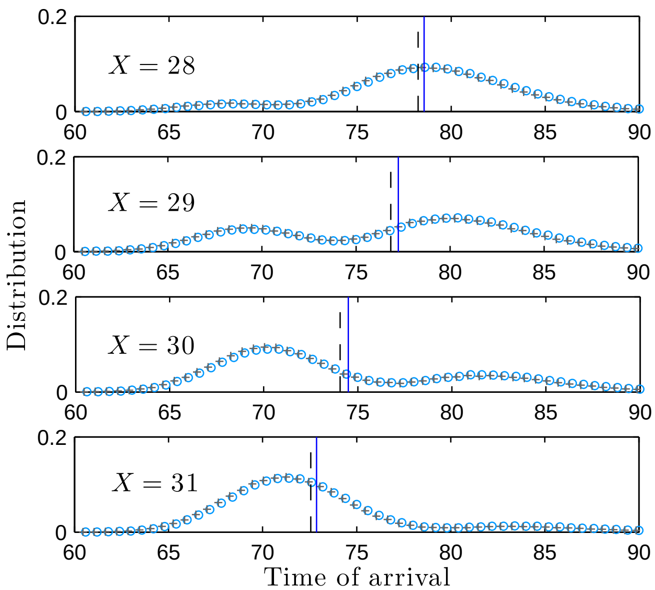

FIG. 4: The time of arrival distributions at the screen positions $X = 28, 29, 30, 31$ ($Y = 600$). Distributions are calculated by the multiple counting scheme ($\circ$) and the first counting scheme ($+$). The solid vertical lines and broken lines represent the averaged multiple counting time and first counting time, respectively.

mechanics while the multiple counting time $\langle T\rangle^{\rm m}_{X,Y}$ coincides with the presence time $\langle T\rangle^{\rm p}_{X,Y}$ calculated by the probability density. The latter result agrees with that of Aoki et al. [10]. It is also found that the time of arrival changes most rapidly in the vicinity of positions giving the intensity minima.

In Fig. 4, we show the time of arrival distributions calculated by both schemes at the screen positions $X = 28, 29, 30, 31$. These positions are located within the vertical lines in Fig. 3, i.e. one of the minimum intensity positions. The time of arrival distributions about other minima behave similarly. From the figure we observe that about the minimum intensity positions two tails of wave packets contribute to the intensity. It is worthwhile to mention that Dalibard et al. observed interferences in the arrival distribution with "slits in time" [17]. Our result suggests that a similar kind of temporal interference will be observed by using ordinary "slits in space" if arrival times of electrons are measured one by one.

In this report, we have shown that the arrival time of electrons in the double slit experiment will have an oscillating behavior, which is, in principle, observable in an experiment. It is shown that the time of arrival is calculated clearly by using stochastic mechanics though there are two schemes for calculation: the first counting scheme and the multiple counting scheme. The former agrees with the arrival time based on Bohmian mechanics while the latter agrees with the presence time derived by a naive consideration in quantum mechanics. However, the numerical results of two schemes differ only slightly.

This work is partially supported by the Ministry of Education, Science, Sports and Culture, Grant-in -Aid for Scientific Research (C).

[1] J. G. Muga, C. R. Leavens, Phys. Rep. **338**, 353 (2000), and references therein.
[2] I. L. Egusquiza and J. G. Muga, Phys. Rev. A **61**, 012104 (1999).
[3] W. Pauli, in: S. Flugge (Ed.), Encyclopedia of Physics, Vol. 5/1, Springer, Berlin, 1958, p. 60.
[4] J. Kijowski, Rept. Math. Phys. **6**, 361 (1974).
[5] N. Grot, C. Rovelli, and R. S. Tate, Phys. Rev. A **54**, 4676 (1996).
[6] J. A. Damborenea, I. L. Egusquiza, G. C. Hegerfeldt, and J. G. Muga, Phys. Rev. A **66**, 052104 (2002).
[7] G. C. Hegerfeldt, D. Seidel, and J. G. Muga, Phys. Rev. A **68**, 022111 (2003).
[8] E. Nelson, Phys. Rev. **150**, 1079 (1966).
[9] F. Guerra, Phys. Rep. **77**, 263 (1981).
[10] K. Aoki, A. Horikosi, and E. Nakamura, Phys. Rev. A **62**, 022101 (2000).
[11] K. Hara, I. Ohba, Phys. Rev. A **62** 032104 (2000).
[12] P. G. Merli. G. F. Missiroli, and G. Pozzi, Am. J. Phys. **44**, 306 (1976).
[13] A. Tonomura, J. Endo, T. Matsuda, T. Kawasaki, and H. Ezawa, Am. J. Phys. **57**, 117 (1989).
[14] K. Yasue and J. C. Zambrini, Ann. of Phys. **159** 304 (1984).
[15] D. de Falco and E. Pulignano, Eur. J. Phys. **9** 32 (1988).
[16] P. R. Holland, *The Quantum Theory of Motion* (Cambridge University Press, Cambridge, 1993).
[17] P. Szriftgiser, D. Guéry-Odelin, M. Arndt, and J. Dalibard, Phys. Rev. Lett. **77** 4 (1996).