Metrizability of GO-spaces and topological groups

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Abstract

In this paper, we give some metrizability theorems* on GO-spaces and topological groups by means of weak topologies or k-networks. Also, we give some related results around GO-spaces and topological groups.

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Introduction

As is well-known, a *linearly ordered topological space* (abbreviated LOTS) is a triple $(X, \mathcal{T}, \leq)$, where $(X, \leq)$ is a linearly ordered (= totally ordered) set, and $\mathcal{T}$ is the order topology by the order $\leq$; that is, $\{(\alpha, +\infty), (-\infty, \alpha) : \alpha \in X\}$ is a subbase for $\mathcal{T}$, here $(\alpha, +\infty) = \{x \in X : x > \alpha\}, (-\infty, \alpha) = \{x \in X : x < \alpha\}$.

A space X is a *generalized ordered space* (abbreviated GO-space) if X is a subspace of a LOTS Y, where the order of X is the one induced by the order of Y; see [15] and [20], for example.

We recall that a space $(X,\mathcal{T})$ is *orderable* if there exists a linear order $\leq$ on X such that the order topology on X given by $\leq$ coincides with the topology $\mathcal{T}$. Obviously, a space X is orderable iff it is homeomorphic to a LOTS. A space X is called *suborderable* if it is homeomorhic to a GO-space. We note that every compact or connected subspace of a suborderable space is orderable.

Examples: (1) As is well-known, the Sorgenfrey line, or the Michael line is a GO-space, but it is not a LOTS with the usual ordering, not even orderable. Also, any Stone-Čech compactification

* Many of these metrizability theorems have been announced in [29], 1999.

$\beta(X)$ of a completely regular, non-countably compact space X is not orderable ([34]). (For a GO-space X, some characterizations for $\beta(X)$ and others to be orderable are given in [32] by means of " cuts " in X).

(2) Let $S = \{0\} \cup (1,2)$ be a subspace of the real line R, and let D be an infinite countable discrete space. Then, S is the topological sum $\{0\} \cup (1,2)$ of LOTS in R. Also, S is an open and closed subset of the product space $S \times D$ which is orderable. However, S is not orderable. (Cf. [34; Remark 5.2]).

(3) A subspace $\{0\} \cup (1,2)$ of R with the usual ordering is not a LOTS, but it is orderable. Also, a space $X = [0, \omega_1]$ obtained by isolating every countable limit ordinal is not a LOTS with the usual ordering, but X is orderable ([15; Example 7.1]).

(4) Let X be the quotient space of the topological sum of three unit intervals $[0,1]$ by identifying the point 0 to a point. Then, X is a union of two closed LOTS', but X is not suborderable. (This is shown directly, or by [5; 6.3.2(b)]).

In this paper, let us say that a space X is a LOTS (resp. GO-space) if X is orderable (resp. suborderable), for it will cause no confusion.

Following [7], a space X *is determined by* a cover $\mathcal{C}$, if $F \subset X$ is closed in X iff $F \cap C$ is closed in C for every $C \in \mathcal{C}$. We use " X is determined by $\mathcal{C}$ " instead of the usual " X has the weak topology with respect to $\mathcal{C}$ ". (For a topics around " weak topologies ", see [28], or [31]).

A space is a *k-space* (resp. *sequential space*) if it is determined by a cover of compact subsets (resp. compact metric subsets). A space is a *quasi-k-space* ([19]) if it is determined by a cover of countably compact subsets.

As is well-known, every k-space (resp. sequential space) is precisely a quotient image of a locally compact space (resp. metric space). Also, every quasi-k-space (resp. k-space) is characterized as a quotient image of an M-space (resp. paracompact M-space); see [19]. Here, a space is an *M-space* if it admits a quasi-perfect map onto a metric space.

A space X *has countable tightness* (or, $t(X) \leq \omega$) if, whenever $x \in clA$, then $x \in clB$ for some countable subset B with $B \subset A$.

It is well-known that $t(X) \leq \omega$ if and only if X is determined by a cover of countable subsets; cf. [17], for example. Sequential spaces are k-spaces of countable tightness, and k-spaces are quasi-k-spaces.

A space X is a *$w\Delta$-space* if there exists a sequence $\{\mathcal{U}_n; n \in N\}$ of open covers of X such that if $x \in X$ and $x_n \in \mathrm{St}(x, \mathcal{U}_n)$, then the sequence $\{x_n\}$ has an accumulation point in X. Every developable space, or every M-space is a $w\Delta$-space.

A space X is a Σ-space (resp. strong Σ-space) [18], if there exist a σ-locally-finite closed cover $\mathcal{F}$, and a cover $\mathcal{C}$ of countably compact closed subsets (resp. compact subsets) in X such that, for $C \subset U$ with $C \in \mathcal{C}$ and U open in X, $C \subset F \subset U$ for some $F \in \mathcal{F}$. Every M-space, or σ-space is a Σ-space. Also, every locally compact GO-space is a Σ-space ([15]).

Let $\mathcal{P}$ be a cover of a space X. Then, $\mathcal{P}$ is a *k-network* for X, if whenever $K \subset U$ with K compact and U open in X, $K \subset \cup\mathcal{P}' \subset U$ for some finite $\mathcal{P}' \subset \mathcal{P}$. Also, $\mathcal{P}$ is a *wcs*-network* [12] for X, if whenever L is a sequence converging to a point $x \in X$ and U is a nbd of x, some $P \in \mathcal{P}$ is contained in U, and contains the sequence L frequently (but, not necessarily contains the point x). Every k-network is clearly a wcs^*-network. For a recent survey on k-networks, see [30], for example.

In this paper, first we make a survey of metrizability theorems around GO-spaces, topological groups, or spaces with certain k-networks. Then, we give metrizability theorems on GO-spaces and topological groups by means of weak topologies or k-networks, and we give some related results. For example, we have the following metrizability theorems (A) and (B).

(A) Every GO-space X with a point-countable wcs^*-network is metrizable if X satisfies one of the following: Locally separable space; $w\Delta$-space; Σ-space.

(B) Every GO-space and topological group G is metrizable if G satisfies one of the following: Locally separable space; Quasi-k-space (in particular, $w\Delta$-space); Σ-space; $t(G) \leq \omega$; Space whose points are G_δ-sets.

We note that, for a topological group G, G is a LOTS if and only if it is a GO-space ([14]).

We assume that spaces are regular T_1, and maps are continuous and onto.

Metrizability theorems

Let us review some metrizability theorems which will be used later on. First, let us recall definitions.

A space X is *bi-sequential* [17] if for every filterbase $\mathcal{F}$ (i.e., any $F \in \mathcal{F}$ is not empty, and for any $A, B \in \mathcal{F}$, there exists $C \in \mathcal{F}$ with $C \subset B \cap C$) in X accumulating to a point $x \in X$, there exists a countable filterbase $\mathcal{G}$ in X converging to x such that $F \cap G \neq \emptyset$ for every $F \in \mathcal{F}$ and $G \in \mathcal{G}$. If we add an prefix " countable " before " filterbase " $\mathcal{F}$ in X, then such a space X is called *countably bi-sequential* (= strongly Fréchet).

Obviously, every first countable space is bi-sequential, and every bi-sequential space is countably bi-sequential.

The *sequential fan* S_ω is the quotient space obtained from the topological sum of countably many convergent sequences by identifying all limit points. For the *Arens' space* S_2, see [6; Example 5.1], [26], [28], etc.

Obviously, any countably bi-sequential space contains no copy of S_ω, and no S_2. Any Fréchet space contains no copy of S_2.

A collection $\mathcal{C}$ in X is *compact-finite* if any compact subset of X meets only finitely many elements of $\mathcal{C}$.

Metrizability theorems: (M1) Every M-space X is metrizable if X has a point-countable base, or X has a σ-space (more generally, X has a G_δ-diagonal).

(M2) Every M-space X with a point-countable k-network is metrizable if X is a paracompact space ([7]), or X is a k-space ([28]).

(M3) Every GO-space X is metrizable if X is a σ-space (more generally, X is a semi-stratifiable space) ([15]).

(M4) Every GO-space X with a G_δ-diagonal is metrizable if X is a $w\Delta$-space or a Σ-space.

(M5) Let X be a k-space with a σ-compact-finite k-network. Then, X is metrizable if X contains no closed copy of S_ω, and no S_2.

(M6) Every bi-sequential topological group G is metrizable ([1]).

(M7) Let G be a topological group which is a k-space with a point-countable k-network. Then, G is metrizable if G contains no closed copy of S_ω, or S_2.

Indeed, (M1) is well-known; see [20; pp.401-408], for example. For (M4), every GO-space with a G_δ-diagonal is paracompact by [15; Theorem 4.5]. Then, if X is a $w\Delta$-space, X is an M-space, thus X is a Σ-space. So, let X be a Σ-space. But, X is a paracompact space with a G_δ-diagonal. Then, X is a σ-space by [18; Theorem 3.15]. Hence, X is metrizable by (M3). For (M5), note that every first countable space with a σ-compact-finite k-network is metrizable by [13]. But, X is a k-space with a point-countable k-network. While, X contains no closed copy of S_ω, and no S_2. Thus, X is first countable by [11; Corollary 3.9]. Hence, X is metrizable. For (M7), G is a topological group, so G contains no closed copy of S_ω iff G contains no closed copy of S_2 by [21; Lemma 2.1]. Hence, G is a k-space with a point-countable k-network, and G contains no closed copy of S_ω, and no S_2. Thus, G is first countable. Hence, G is metrizable by (M6).

Now, let us give metrizability theorems for GO-spaces or topological groups. The following metrizability theorem for GO-spaces holds.

Theorem 1. Let X be a GO-space. If X has a point-countable wcs^*-network, then the following (1) and (2) hold.

(1) Suppose that (a), (b), or (c) below holds. Then, X is a paracompact space having a point-countable base.

In particular, if X has a σ-compact-finite wcs^*-network, then X is metrizable.

(a) all points of X are G_δ-sets.

(b) X is a quasi-k-space.

(c) $t(X) \leq \omega$.

(2) Suppose that (a), (b), or (c) below holds. Then, X is metrizable.

(a) X is locally separable.

(b) X is a (locally) $w\Delta$-space.

(c) X is a (locally) Σ-space.

Not every LOTS X with a compact-finite k-network of singletons is first countable (indeed, let X be the space Z in [15; Example 7.1]).

Note that not every countably compact space with a point-countable k-network is metrizable ([7]). But, among GO-spaces or k-spaces, every countably compact space (more generally, M-space) with a point-countable k-network is metrizable in view of the previous theorem and (M2).

Not every LOTS X with a σ-point-finite base is metrizable (indeed, let M be the Michael Line, and let M^* be the ordinary LOTS containing a closed copy of M; here, for a GO-space X, the LOTS X^* is defined in [15; Definition 2.5] or [20; p.457]. Hence, M^* is a desired one). Also, not every LOTS with a locally countable network is metrizable by a LOTS $[0, \omega_1)$. But, the following holds.

Corollary 1.1. Every GO-space with a σ-locally countable wcs^*-network is metrizable.

Let X be a space, and let $\{X_n : n \in N\}$ be an increasing (not necessarily closed) cover. By a *direct* (or *inductive*) *limit* $X = \varinjlim\{X_n : n \in N\}$, we mean X is determined by the cover $\{X_n : n \in N\}$.

Let us recall that a (non-empty) space X has *large inductive dimension zero* (i.e., $Ind\ X = 0$) if for every closed set $F \subset X$ and any open set U with $F \subset U$, there exists an open and closed subset A of X with $F \subset A \subset U$. If we replace " closed set " by " single point ", then X has *small inductive dimension zero* (i.e., $ind\ X = 0$). For the definitions of covering dimension zero (i.e., $dim\ X = 0$), see [5], for example. Obviously, $Ind\ X = 0$ implies $ind\ X = 0$. As is well-known, for a normal space X, $Ind\ X = 0$ and $dim\ X = 0$ are equivalent, and when X is Lindelöf, $Ind\ X = 0$, $ind\ X = 0$, and $dim\ X = 0$ are equivalent.

In this paper, for a *normal space* X, let us call X is *zero-dimensional* (resp. *strong zero-dimensional*), if $ind\ X = 0$ (resp. $Ind\ X = 0$) in view of [5]. We recall that there exists a zero-dimentional metric space which is not strongly zero-dimentional. But, for a separable metric space X, X is zero-dimentional if and only if it is strongly zero-dimentional.

Corollary 1.2. Let $X = \varinjlim\{X_n : n \in N\}$ such that X_n are metric subspaces. Then, the following hold.

(1) If X is a GO-space, then X is a metrizable space.
(2) If each X_n is a strongly zero-dimensional space, then the following are equivalent.
(a) X is a GO-space.
(b) X is a LOTS.
(c) X is a strongly zero-dimensional metrizable space.

In particular, if each X_n is locally separable, zero-dimensional, then we can replace (c) by " X is the topological sum of subspaces of the Cantor set 2^ω ".

Not every compact topological group is metrizable; see Remark 3. But, among LOTS', every locally compact topological space is metrizable; see [34], etc.

Among GO-spaces, the following theorem holds for topological groups.

A space is *totally connected* (= hereditarily disconnected [5]) if every component is one point. Every zero-dimensional space is totally disconnected. A space is a *P-space* if all G_δ-sets are open.

Theorem 2. Let G be a topological group. If G is a GO-space, then G is a metrizable, or a totally disconnected P-space.

We note that every topological group which is a LOTS need not be metrizable; see Remark 2. But, the following holds.

Corollary 2.1. Let G be a topological group which is a GO-space. Then, G is a metrizable if one of the following properties (a) $\sim$ (g) holds.
(a) G is locally separable.
(b) all points of G are G_δ-sets.
(c) G is a quasi-k-space.
(d) $t(G) \leq \omega$.
(e) G is a (locally) Σ-space.
(f) some countably compact subset of G is infinite.
(g) some connected subset of G contains at least two points.

Corollart 2.2. Let $G = \underrightarrow{\lim}\{G_n : n \in N\}$ such that G_n are locally compact, topological groups. Then, the following (a), (b) and (c) are equivalent.
(a) G is a GO-space.
(b) G is a LOTS.
(c) One of the following properties holds: G is a discrete space; G is the topological sum of the real lines R; and G is the topological sum of the Cantor sets.

A space X is *biradial* [1] if for every filterbase $\mathcal{F}$ in X accumulating to a point $x \in X$, there exists a chain $\mathcal{C}$ (i.e., $\mathcal{C}$ is a filterbase such that for every $A \in \mathcal{C}$ and every $B \in \mathcal{C}$, $A \subset B$ or $B \subset A$) in X converging to x such that $A \cap B \neq \emptyset$ for every $A \in \mathcal{F}$ and $B \in \mathcal{C}$.

Every subspace of biradial space is biradial. Every bi-sequential space, or every GO-space is biradial.

A map $f : X \to Y$ is *bi-quotient* [16] if, whenever $y \in Y$ and $\mathcal{U}$ is a cover of $f^{-1}(y)$ by open subsets of X, then finitely many $f(U)$, with $U \in \mathcal{U}$, cover a nbd of y in Y.

Every bi-sequential space is characterized as a bi-quotient image of a metric space ([17]).

Every biradial space is characterized as a bi-quotient image of a LOTS in view of [9; Theorem 2.8]*.

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* M. Sakai gave a modification of the proof of this result.

Obviously, not every locally compact, separable metric GO-space is a LOTS. But, for topological groups, we have the following due to [14].

Theorem 3. Let G be a topological group. Then the following (a) and (b) are equivalent. When G is not metrizable, the following (a), (b), and (c) are equivalent.

(a) G is a LOTS.

(b) G is a GO-space.

(c) G is a biradial space.

Related to the previous theorem, note that every metric (hence, biradial) GO-space need not be a LOTS. Also, note that every metric topological group need not be a GO-space (in view of Remark 4(1)). While, among totally disconnected metric spaces, GO-spaces and LOTS are equivalent by Lemma 8(3), but we can't replace " metric " by " first countable " (by the Sorgenfrey line).

Every biradial topological group is hereditarily collectionwise normal by [1], more precisely, the following holds.

Corollary 3.1. For a topological group G, if G is a biradial (in particular, GO-space), then G is hereditarily paracompact.

Lemmas

The following holds in view of the proof of [27; Proposition 1.2(1)].

Lemma 1. Let $\mathcal{P}$ be a point-countable wcs^*-network for a space X. Then, for $K \subset U$ with K sequentially compact and U open in X, $K \subset \cup\mathcal{P}' \subset U$ for some finite $\mathcal{P}' \subset \mathcal{P}$

Lemma 2. Every sequentially compact space X with a point-countable wcs^*-network is metrizable.

Proof. We refer to the proof of [2; Theorem 3.1], where X is compact. Let $x, y \in Y$ with $x \neq y$. Then, there is a nbd V_x of x with $clV_x \subset X - \{y\}$. Let $\mathcal{P}$ be a point-countable wcs^*-network for X. Since clV_x is sequentially compact, and $U = X - \{y\}$ is open in X, by Lemma 1, there exists a finite $\mathcal{P}' \subset \mathcal{P}$ with $clV_x \subset \cup\mathcal{P}' \subset U$, so that $x \in int \cup \mathcal{P}'$, but $y \notin \cup\mathcal{P}'$. Also, the cover $\mathcal{R}$ in the proof of [2; Theorem 3.1] still have a finite subcover even if X is countably compact. Indeed, let $\mathcal{R}^* = \{\cup\mathcal{F} : \mathcal{F} \subset \mathcal{R}$ is finite$\}$. Then, X is covered by $\{intR : R \in \mathcal{R}^*\}$, thus, X is determined by this open cover. Then, X is determined by $\mathcal{R}^*$. Hence, X is a countably compact space, and $\mathcal{R}$ is a point-countable cover of X such that $\{\cup\mathcal{F} : \mathcal{F} \subset \mathcal{R}$ is finite$\}$ determines X. Then, since X is countably compact, $\mathcal{R}$ has a finite subcover by [7; Proposition 2.1]. Thus, as is seen in the proof of [2; Theorem 3.1], X has a countable network. Since X is countably compact, X is compact. Thus, X is metric.

The following is well-known, and it is shown routinely.

Lemma 3. Let X be a GO-space. Then the following (1) and (2) hold.

(1) X is first countable if (a), (b), or (c) below holds.

(a) X is separable.

(b) all points of X are G_δ-sets.

(c) $t(X) \leq \omega$.

(2) Every countably compact subset of X is sequentially compact.

The following holds by Lemmas 1 and 3(2).

Lemma 4. Every point-countable wcs^*-network for a GO-space is a k-network.

The following holds by [7; Corollary 3.6], and [27; Corollary 1.5] (or Lemma 1).

Lemma 5. Every first-countable space with a point-countable wcs^*-network has a point-countable base.

Lemma 6. Every paracompact space X with a σ-locally countable network $\mathcal{P} = \cup\{\mathcal{P}_n : n \in N\}$ is a σ-space.

Proof. For $n \in N$, and $x \in X$, let $V_{x,n}$ be a nbd of x such that $V_{x,n}$ meets only countably many elements of $\mathcal{P}_n$. Since X is paracompact, X has a locally finite refinement $\mathcal{U}_n = \{U_\alpha : \alpha \in \Lambda\}$ of $\{V_{x,n} : x \in X\}$. For $U_\alpha \in \mathcal{U}_n$, let $U_\alpha \cap \mathcal{P}_n = \{P_{n,i}^{(\alpha)} : i \in N\}$. Let $\mathcal{W}_{n,i} = \{P_{n,i}^{(\alpha)} : \alpha \in \Lambda\}$, and let $\mathcal{W}_n = \cup\{\mathcal{W}_{n,i} : i \in N\}$. Then, $\mathcal{W} = \cup\{\mathcal{W}_n : n \in N\}$ is a σ-locally finite network for X. Hence, X is a σ-space.

Lemma 7. For a topological group G, if G is a biradial space, then G is metrizable, or a P-spacce ([1]).

Lemma 8. (1) For a GO-space X, the following are equivalent.
(a) X is zero-dimensional.
(b) X is strongly zero-dimensional.
(c) X is totally disconnected.
(2) Every strongly zero-dimensional metric space is a LOTS.
(3) Every metric, totally disconnected GO-space is a LOTS.

Proof. For (1), (c) $\to$ (b) is due to [23], here the implication for X being a LOTS is due to [8]; see [5; 6.3.2]. Indeed, let X be a totally disconnected GO-space. Then, the LOTS X^* (in [15; Definition 2.5]) is totally disconnected. Then, X^* is strongly zero-dimensional. But, X is a closed subset of X^*. Thus, X is strongly zero-dimensional. (2) is due to [8]; see [5; 6.3.2]. (3) holds by (1) and (2).

Proofs of Theorems and Corollaries

Proof of Theorem 1: Every countably compact subset of X is metrizable by Lemmas 2 and 3(2). Thus, for (1), if X is a quasi-k-space, X is sequential, thus $t(X) \leq \omega$. Then, for any case (a), (b), or (c), X has a point-countable base by Lemmas 3(1) and 5. Thus, X is meta-Lindelöf. But, X is a GO-space. Then, X is paracompact by [15; Theorem 4.2]. For the latter part of (1), X is a first countable space. Besides, X has a σ-compact-finite k-network by Lemma 4. Then, X is metrizable by (M5). Next, we prove (2). For (a), let X be separable. Then, X has a point-countable base by Lemmas 3(1) and 5. But, X is separable. Thus, as is well-known, X is (separable) metrizable. For (b), let X be a $w\Delta$-space. Then, X is a quasi-k-space in view of [19] (or, each $x \in X$ is contained in a countably compact closed subset which is a G_δ-set in X, thus, x is a G_δ-set, which implies X is first countable by Lemma 3(1)). Thus, X is a paracompact space with a point-countable base by (1). Then, X is an M-space, for X is a paracompact $w\Delta$-space. But, X has a point-countable base. Then, X is metrizable by (M1). For (c), X is a strong Σ-space, and it has a point-countable k-network by Lemma 4. Thus, X is a σ-space by [7; Corollary 3.8]. Then, X is metrizable by (M3). Finally, if X

has the property in (a), (b), or (c) locally, then X is locally metric. Then, since X is first countable, X is paracompact. Hence, X is also metrizable.

Proof of Theorem 2: Suppose that G is not metrizable. Since every GO-space is biradial, by Lemma 7, G must be a P-space. On the other hand, if G is not totally disconnected, G must be metrizable (actually, G is the topological sum of the real lines R) in view of the proof of Theorem 2.4 in [34] under G being a GO-space. Thus G must be totally disconnected. Hence, G is a totally disconnected P-space.

Proof of Theorem 3: The proof is given in [14], so we shall omit it (Theorem 3 can be shown in view of results in [1], [22], and [34], using Lemma 8(3)).

Proofs of Corollaries: For Corollary 1.1, note that each point of X is a G_δ-set. Then, by Theorem 1(1), X is a paracompact space with a point-countable base. Since X is paracompact, X is a σ-space by Lemma 6. Since X is a GO-space, X is metrizable by (M3).

For Corollary 1.2, since X is determined by a countable cover of metric subspaces X_n, X is a sequential space. Also, X has a point-countable wcs^*-network, because every convergent sequence in X is frequently in some X_n containing its limit point. Thus, for (1), if X is a GO-space, X is a paracompact first countable space by Theorem 1. Since X is a first countable space determined by $\{X_n : n \in N\}$, it is routine that each point of X has a nbd V_x which is contained in some X_n, hence V_x is metric. Hence, X is locally metric. But, X is a paracompact space. Then, as is known, X is metrizable. For (2), let X be a GO-space. Then, X is metrizable. Thus, X has a locally finite closed cover of strongly zero-dimensional spaces. Thus, X is a strongly zero-dimension metrizable space, hence, X is a LOTS by Lemma 8(2). For the latter part, if X is a GO-space, X is locally separable metrizable, thus, X is the topological sum of zero-dimensional, separable metrizable subspaces. But, as is well-known, these zero-dimensional, separable metrizable subspaces are subsets of the Cantor set. Hence, X is also a LOTS by Lemma 8(2).

For Corollary 2.1, let G be not metrizable. Then, G is a P-space by Theorem 2. Thus, obviously every countably compact subset of G is finite which is contradictory to (f). Also, if G satisfies one of properties (a) $\sim$ (e), it is routine that G is a discrete space as is pointed out by [25; Proposition 1.3], a contradiction. While, G is also totally disconnected by Theorem 2, which is contradictory to (g). Hence, G must be metrizable if G satisfies one of the properties (a) $\sim$ (g).

For Corollary 2.2, as is well-known, G is a topological group since G is determined by an increasing cover $\mathcal{G} = \{G_n : n \in N\}$ of locally compact subsets. While, G is a k-space, because G is determined by the cover $\mathcal{G}$ of locally compact subspaces. Let G be a GO-space. Then, G is metrizable by Corollary 2.1, hence G is first countable. Thus, G is locally compact, because G is determined by the decreasing countable cover $\mathcal{G}$ of locally compact subspaces. Hence, G is a locally compact topological group which is a GO-space. Then, the property (a), (b), or (c) holds in view of Theorems 2.4 and 2.5 in [34] under G being a GO-space.

For Corollary 3.1, if G is metrizable, then it is hereditarily paracompact. So, assume that G is not metrizable. Then, G is a LOTS by the latter part of Theorem 3. Thus, since G is a topological group, G is hereditarily paracompact by [22; Theorem 8].

Remarks

Remark 1. Let us give a general form of Theorem 1 and Corollary 1.1 as follows: Let X be a GO-space. Let $Y \subset X$, and let $Y = \cup\{Y_n : n \in N\}$, where each Y_n is a closed subspace of Y. Then, the following (1) and (2) hold.

(1) Let X have a point-countable wcs^*-network. If each Y_n satisfies one of (a), (b), and (c) in Theorem 1(1) (resp. Theorem 1(2)), then Y has a point-countable base (resp. Y is metrizable). In

particular, if X has a σ-compact finite k-network, then Y is metrizable.

(2) If each Y_n has a σ-locally countable wcs^*-network, then, Y is metrizable.

Indeed, note that every subset of a GO-space is a GO-space; and every GO-space X which is a countable union of closed first countable (resp. closed metric) subspaces is first countable (resp. metrizable), because each point of X is a G_δ-set (resp. X is a σ-space), thus, X is first countable by Lemma 3(1) (resp. X is metrizable by (M3)). Then, (1) holds in view of Theorem 1, and Lemma 5. Similarly, (2) holds in view of Corollary 1.1.

Remark 2. There exists a LOTS and topological group G which is a totally disconnected P-space, but G is not metrizable (not of countable tightness, nor a quasi-k-space)*.

Indeed, let D be an infinite abelian group with the discrete topology. Let $G = D^{\omega_1}$. For each $f \in G$, and $\alpha \in \omega_1$, let $W(f;\alpha) = \{g \in G : f(\beta) = g(\beta)$ for any $\beta < \alpha\} \cup \{f\}$. Let $\mathcal{T}$ be a topology having a base $\{W(f;\alpha) : f \in G,\ \alpha \in \omega_1\}$. The, it is easy to show that the topology $\mathcal{T}$ contains the usual product topology, and each basic nbd $W(f;\alpha)$ is an open and closed set which is a G_δ-set in the product topology. Also, $\mathcal{T}$ is equivalent to the order topology induced by a linear order $<$ on G as follows: $f < g$ iff for some $\alpha \in \omega_1$, $f(\beta) = g(\beta)$ for any $\beta < \alpha$, and $f(\alpha) < g(\alpha)$. Thus, G is a topological group which is a LOTS and P-space. But, the character of G is obviously ω_1, thus, G is not metrizable. Thus, X is not of countable tightness, nor a quasi-k-space by Corollary 2.1.

Let us recall that a space X is *isocompact* if every *closed* countably compact subset of X is compact, and that a space X is *hereditarily isocompact* if every countably compact subset of X is compact. As is well-known, every meta-Lindelöf space is isocompact. Note that every isocompact space is hereditarily isocompact if it is sequential, or all points are G_δ-sets (because, every countably compact subset is closed).

Remark 3. (1) (i) There exists a compact connected topological group G, but G doesn't satisfy any of the following properties.

(a) hereditarily normal.

(b) hereditarily isocompact.

(c) biradial space.

(d) space with a point-countable k-network.

(e) space having countable tightness.

(ii) There exists a countably compact, countably bi-sequential topological group Σ, but Σ doesn't satisfy any of the properties (a) $\sim$ (d) in (i), here we can replace (b) by (b)′ isocompact.

(2) (i) Every (locally) compact topological group G is metrizable if G satisfies one of the properties (a) $\sim$ (e) in (i).

(ii) (CH) There exists a countably compact group satisfying (a), (d), and (e), but it is not metrizable.

Proof. We show (1) holds. First, let us recall a well-known concept. Let A be an uncountable set, and for each $\alpha \in A$, let X_α be a space containing at least two points p_α and q_α. Let $X = \Pi\{X_\alpha : \alpha \in A\}$, and let Σ be a Σ-*product* of X_α's; that is, $\Sigma = \{x = (x_\alpha) \in X : x_\alpha \neq p_\alpha$ for at most countably many $\alpha \in A\}$ be a subspace of X. Then the the following facts hold.

(F_1) If each X_α is compact (resp. bi-sequential), then Σ is countably compact (resp. countably bi-sequential).

(F_2) Σ doesn't satisfy any of the properties (a) $\sim$ (d) in (i), here we can replace (b) by the property (b)′.

(F_3) The product $\{0,1\}^{\omega_1}$ doesn't satisfy the property (e) in (i).

* This remark is a personal commumication from M. Sakai.

Indeed, (F_1) is known or routinely shown. For the parenthetic part, Σ has countable tightness by [10; Proposition 1], because every subset of bi-sequential space is bi-sequential, and every countable product of bi-sequential spaces is bi-sequential by [17; Proposition 3.D.3]. Then, any countable subset of Σ is bi-sequential, thus, Σ is countably bi-sequential by [17; Proposition 8.7]. For (F_2), let $C = \Pi\{S_\alpha : \alpha \in A\}$, here $S_\alpha = \{p_\alpha, q_\alpha\}$. Let $\Sigma^* = \Sigma \cap C$. Then, Σ^* is a Σ-product of S_α's. Then, by [4; Proposition 2], Σ^* is not hereditarily normal. On the other hand, by (F_1), Σ^* is a countably compact subspace which is closed in Σ. But, Σ^* is not compact, because Σ^* is a dense proper subset of C. This implies that Σ is not isocompact, and it has no point-countable k-networks by *Theorem A*. To show Σ is not a biradial space, let Σ be biradial, then so is Σ^*. But, Σ^* is considered as a topological group, because so is each S_α. Then, Σ^* is paracompact by Corollary 3.1. Hence, Σ^* is compact, a contradiction. Thus, Σ is not a biradial space. The fact (F_3) holds in view of the proof of [24; Lemma 2.9].

Now, let us show (1) holds. For (i), let $G = X^{\omega_1}$, where X is the circle S^1. Then, G is a compact connected topological group. We note that G contains a subspace $Q = P^{\omega_1}$, here P consists of two points. Also, Q contains the Σ-product of P's. Thus, G is a desired one by (F_2) and (F_3). For (ii), let each X_α be a topological group, and let p_α be the identity element of X_α. Then, Σ is a topological group. Thus, if each X_α is a compact and metrizable topological group, then, Σ is a desired one in view of (F_1) and (F_2).

Finally, we show (2) holds. For (i), the property (c) implies (a) by Corollary 3.1. Also, for the property (d), G is metrizable by (M7). For properties (a), (b) and (e), first we assume that G is compact. We recall Šapirovskiĭ theorem (see [3; Theorem 1.45], for example) that every compact topological group G adomits a map f from G onto $Y = [0,1]^\alpha$, here $\alpha = w(G)$. Since f is a perfect map, Y satisfies the propertiy (a), (b), or (e) if so does G respectively. While, Y contains a closed subset $C = \{0,1\}^\alpha$. But, if $\alpha = \omega_1$, C satisfies none of (a), (b), and (e). Then, $\alpha \leq \omega$ if G satisfies (a), (b), or (e), hence G is metrizable. Now, let G be locally compact. Then, the identity e in G is contained in a compact subgroup H which is a G_δ-subset of G. (In fact, there exist open nbds V_n $(n \in N)$ of e such that $V_n V_n{}^{-1} \subset V_{n+1}$, $clV_n \subset V_{n+1}$, and clV_1 is compact. Put $H = \cap\{V_n : n \in N\}$). Since H is first countable by the above, e is a G_δ-set in G. Then, each point of G is a G_δ-set in G. But, G is locally compact. Thus, G is first countable. Hence, G is metrizable by (M6). For (ii), the (hereditarily separable) topological group G in [33; Theorem 1] is a desired one under (CH). Furthermore, every compact subset of G is finite*, hence G has a point-countable k-network of singletons.

Remark 4. (1) For the product space $X \times Y$ of biradial spaces (or GO-spaces) X and Y, the following (i) and (ii) hold by [14], but for the parenthetic part in (i), note that if $X \times Y$ is a GO-space, then X is discrete, or Y is totally disconnected in view of [34; Proposition 2.3].

(i) Let $X \times Y$ be a biradial space (resp. GO-space). Then the following (a), (b), or (c) holds.

(a) X or Y is discrete.

(b) X and Y are bi-sequential spaces (resp. totally disconnected, first countable spaces).

(c) X and Y are P-spaces (resp. totally disconnected P-spaces).

(ii) X^ω is biradial if and only if X is bi-sequential. If X^ω is a GO-space, then X is a totally disconnected, first countable.

(2) For a strongly zero-dimensional metric space X (resp. zero-dimensional, separable metric space X) with X containing at least two points, X^α is a biradial space $\Longleftrightarrow$ X^α is a GO-space $\Longleftrightarrow$ X^α is a LOTS $\Longleftrightarrow$ X^α is a strongly zero-dimensional metric space (resp. X^α is a subspace of the Cantor set), and $\alpha \leq \omega$.

(3) There exists a totally disconnected GO-space X such that any finite product X^n is biradial, but, X is not of countable tightness, also, X^ω is not biradial.

Proof. We show (2) holds. Let $Y = X^\alpha$ be a biradial space. Suppose that $\alpha \geq \omega_1$. Then, Y

* This is a personal communication from M. G. Tkachenko.

contains a copy of $S = \{0,1\}^{\omega_1}$. But, S is a biradial space which is homeomorphic to $S \times \{0,1\}^{\omega}$. Then, S is bi-sequential by (i) in (1), so $t(S) \leq \omega$. This is a contradiction by the last part of the proof of Remark 3. Thus, $\alpha \leq \omega$. While, the property " strongly zero-dimensional metric space " is countably productive; see [5], for example. Then, $Y = X^{\alpha}$ is a strongly zero-dimensional metric space, and $\alpha \leq \omega$. Thus, Y is a LOTS by Lemma 8(2). Finally, we show (3) holds. Let $X = [0, \omega_1]$ be a space obtained from the usual order topology by isolating every countable limit ordinal. Then, X is a totally disconnected GO-space which is not of countable tightness. For each $n \in N$, X^n is biradial, because every point $p \in X^n$ has the obvious local base η which is decreasing (hence, η is a chain converging to the point p). But, X^{ω} is not biradial by (ii) of (1).

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