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Direct Sum Representation of Torsion Free Injective Modules

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Abstract

Let R be a ring and τ a hereditary torsion theory which is cogenerated by an injective left R-module. Then, every τ-torsion free module is embedded in a direct sum of τ-finitely generated τ-torsion free left R-module, if and only if R is τ-Artinian and there exists a τ-finitely generated injective left R-module E such that τ is cogenerated by E.

Key words: hereditary torsion theory, τ-torsion free modules, injective modules, τ-Artinian rings, Morita duality

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Let R be a ring with identity. In [2] Faith-Walker proved that if every left R-module is embedded in a direct sum of finitely generated left R-modules, then R is left Artinian. It is to be noted that this condition is a necessary and sufficient condition for R to be an left Artinian ring with Morita duality [11], *i.e.*, R is left Artinian and there exists a finitely generated injective cogenerator of the category of the left R-modules. In this paper we generalize this Faith-Walker's result to the situation of hereditary torsion theories ([3], [12]).

Let τ be a hereditary torsion theory cogenerated by an injective left R-module E. A left R-module M is called τ*-torsion free*, if M is embedded in a direct product of copies of E. On the other hand, M is said to be τ*-torsion*, if Hom $({}_RM, {}_RE) = 0$. We shall say that a submodule N of M is τ*-closed* (resp. τ*-dense*), if M/N is τ-torsion free (resp. τ-torsion). A ring R is τ*-Artinian* (resp. τ*-Noetherian*), if R satisfies DCC (resp. ACC) on τ-closed left ideals. If S is a set, the cardinal number of S is denoted by card S. Let S be an index set such that *card* $S = c$. Then, a left R-module M is said to *c-generated,* if M is a epimorphic image of the free left R-module $\oplus_{s\in S}R^{(s)}$ where $R^{(s)}$ is a copy of R. Therefore, if M has a τ-dense (resp. essential) *c-generated* submodule, then M is said to be a τ*-c-generated* module (resp. *essentially c-generated module*). Especially, if M has a finitely generated submodule N such that M/N is τ-torsion, then M is said to be τ*-finitely generated*.

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In Theorem 4, generalizing the above result of Faith-Walker, we show that for a ring R every τ-torsion free left R-module is embedded in a direct sum of τ-finitely generated τ-torsion free modules, if and only if R is τ-Artinian and τ is cogenerated by a τ-finitely generated injective left R-module.

Throughout this paper R denotes a ring with identity and τ denotes a hereditary torsion theory cogenerated by an injective left R-module. Every homomorphism between modules will be written on the opposite side of scalars.

Proposition 1. *If every τ-torsion free injective left R-module is a direct sum of essentially c-generated submodules, then R is τ-Noetherian.*

Proof. We only check that the proof of [1, Theorem 25. 8] is also valid in our case. Assume τ is cogenerated by an injective left R-module E. We may assume that *card* $R \leq c$ and *card* $E \leq c$. Let B be a set such that *card* $B>2^c$. Then, $\Pi_{b\in B} E^{(b)} = \oplus_{\alpha\in A} M_\alpha$, where $E^{(b)}$ is a copy of the left R-module E and M_α has a c-generated essential submodule N_α. Let E_1 be a monomorphic image of E in $\oplus_{\alpha\in A} M_\alpha$. Then, there exists a subset A_1 of A such that $E_1 \subseteq \oplus_{\alpha\in A_1} M_\alpha$ and card $A_1 \leq c$. Let $i_b : E \to \Pi_{b\in B} E^{(b)}$ be the natural injection. Suppose $(\oplus_{\alpha\in A_1} M_\alpha) \cap (E)i_b \neq 0$ for every $b\in B$. Since $\oplus_{\alpha\in A_1} M_\alpha$ is an essential extension of $\oplus_{\alpha\in A_1} N_\alpha$, we have $(\oplus_{\alpha\in A_1} N_\alpha) \cap (E)\, i_b \neq 0$. Then,

$$card\ B \leq card\ \oplus_{b\in B}((\oplus_{\alpha\in A_1} N_\alpha) \cap (E)i_b) \leq 2^c < card\ B.$$

This is a contradction. Hence there exists $b\in B$ such that $(\oplus_{\alpha\in A_1} N_\alpha)\cap(E)i_b = 0$ and hence $(\oplus_{\alpha\in A_1} M_\alpha)\cap (E)i_b = 0$. Therefore, it is evident that E is isomorphic to a submodule U_2 of $\oplus_{\alpha\in A\setminus A_1} M_\alpha$. Consequently, there exists a subset A_i of $A\setminus A_1\cup\cdots\cup A_{i-1}$ such that U is isomorphic to a submodle E_{i+1} of $\oplus_{\alpha\in A_i} M_\alpha$ (i=2, 3, $\cdots$). Hence the direct summand $\oplus_{i\in N} E_i$ of $\Pi_{b\in B} E^{(b)}$ is injective, where N is the set of natural numbers. Thus, R satisifies ACC on annihilator left ideals of subsets of E, i. e., R is τ-Noetherian.

Lemma 2. *Assume that M is a τ-torsion free left R-module and $M = \oplus_{\alpha\in A} M_\alpha$. Let L be a submodule of M. If there exists a infinite cardinal number c such that L is τ-c-generated, there exists a subset B of A such that card $B \leq c$ and $L \subseteq \oplus_{\alpha\in B} M_\alpha$.*

Proof. Let K be a τ-dense submodule of L. For every $x \in K$ there exists a finite subset F_x of A such that $x \in \oplus_{\alpha\in F_x} M_\alpha$. Therefore, we have $K \subseteq \oplus_{\alpha\in B} M_\alpha$, where $B = \cup_{x\in K} F_x$. Suppose that there exists $y \in L$ such that y is not contained in $\oplus_{\alpha\in B} M$. Then, we can select an $\alpha \in A$ so that there exists a canonical R-homomorphism $f : L \to M_\alpha$ such that $(y)f \neq 0$ and $(K)f = 0$. Since L/K is τ-torsion, there exists a τ-dense left ideal D such that $Dy \subseteq K$. This is a contradiction, since $0 \neq (Dy)f$. Then, $L \subseteq \oplus_{\alpha\in B} M$ and this completes the proof, since *card* $B \leq c$.

Let c be a cardinal number. Then, it is easily checked that every homomorphic image of a τ-c-generated left R-module is τ-c-generated. The following result is originally based on [4].

Lemma 3. *Let c be an infinite cardinal. If M is a direct sum of τ-c-genereated τ-torsion free left R-module, then so is every direct summand of M.*

Proof. Like as Proposition1 we check that the proof of [1, Theorem 26. 1] is valid in our case. Put $M = \oplus_{\alpha\in A} M_\alpha$, where M_α is a τ-c-genereated τ-torsion free left R-module. Assume $M = K \oplus L$. Let $e : M \to K$ and $f : M \to$ L be canonical projections. Let $\{K_\beta\}_{\beta\in B}$ (resp. $\{L_\gamma\}_{\gamma\in C}$) be the set of all τ-c-generated submodules of K (resp. L). Let Σ be the set of triples (A', B', C') such that $A' \subseteq A$, $B' \subseteq B$, $C' \subseteq C$ and $\oplus_{\alpha\in A'} M_\alpha = \oplus_{\alpha\in B'} K_\alpha + \oplus_{\alpha\in C'} L_\alpha$. Then, Σ is a partial ordered set defining $(A', B', C') \leq (A'', B'', C'')$ in case $A' \subseteq A''$, $B' \subseteq B''$, $C' \subseteq C''$. Let (A', B', C') be a maximal element of Σ. Suppose $A' \neq A$ and $\alpha \in A\setminus A'$. Since $(M_\alpha)e$ and $(N_\alpha)f$ are homomorphic images of τ-c-generated left R-modules, we can deduce (M_α)e + (N_α)f is τ-c generated. Therefore, by Lemma 2 there exists a subset D_1 of A such that $card\ D_1 \leqq c$ and $(M_\alpha)e + (N_\alpha)f \subseteq \oplus_{\alpha \in D_1} M_\alpha$. Therefore, there exists a subset D_i ($i = 2, 3, \cdots$) such that $D_{i-1} \subseteq D_i$ and $(\oplus_{\alpha \in D_{i-1}} M_\alpha)\, e \oplus (\oplus_{\alpha \in D_{i-1}} M_\alpha)\, f \subseteq \oplus_{\alpha \in D_i} M_\alpha$. Set $D = \cup_{i \in N} D_i$, where N is the set of natural numbers. Then, $(\oplus_{\alpha \in D} M_\alpha)\, e \oplus (\oplus_{\alpha \in D} M_\alpha)\, f = \oplus_{\alpha \in D} M_\alpha$, and this implies $(\oplus_{\alpha \in D \cup A'} M_\alpha) e \subseteq \oplus_{\alpha \in D \cup A'} M_\alpha$. It follows that $\oplus_{\alpha \in B'} K_\alpha$, which is a direct summand of $\oplus_{\alpha \in D \cup A'} M_\alpha$, is a direct summand of $(\oplus_{\alpha \in D \cup A'} M_\alpha)\, e$. Hence there exist submodules P and Q of M such that

$$(\oplus_{\alpha \in B'} K_\alpha) \oplus P = (\oplus_{\alpha \in D \cup A'} M_\alpha) e \text{ and } (\oplus_{\alpha \in C} L_\alpha) \oplus Q = (\oplus_{\alpha \in D \cup A'} M_\alpha) f.$$

This implies $P \oplus Q$ is isomorphic to $\oplus_{\alpha \in D \setminus A'} M_\alpha$. Then, P and Q are τ-c-generated, since they are homomorphic images of τ-c-generated modules. Hence there exists $\beta \in B'$ and $\gamma \in C'$ such that $P = K_\beta$ and $Q = L_\gamma$. Thus, we have $(A' \cup \{\alpha\}, B' \cup \{\beta\}, C' \cup \{\gamma\}) \in \Sigma$ and this is a contradiction. This completes the proof.

In the following let us denote $I(M)$ the injective hull of the left R-module M. A τ-torsion free left R-module M is called τ*-cocritical*, if M/N is τ-dense for every non-zero submodule N of M. Let T be a τ-torsion ideal of R. If R is τ-Noetherian and $I(R/T)$ is τ-finitely generated, then R is τ-Artinian (see [5], [7]). Now, we are able to prove the following

Theorem 4. *The following conditions are equivalent for a ring R.*

(*i*) *Every τ-torsion free left R-module is embedded in a direct sum of τ-finitely generated τ-torsion free left R-modules.*

(*ii*) *R is a τ-Artinian ring and there exists a τ-finitely generated injective left R-module E such that τ is cogenerated by E.*

Proof. (i) $\Rightarrow$ (ii). Let M be an arbitrary τ-torsion free injective left R-module. Then, M is a direct summand of a direct sum of τ-finitely generated τ-torsion free left R-modules. Then, by Lemma 3 M is a direct sum of τ-countably generated left R-module. It follows that R is τ-Noetherian from Proposition 1. Let T be the τ-torsion ideal of R. Then, $I(R/T)$ is embedded in $\oplus_{\alpha \in A} M_\alpha$, where M_α is a τ-finitely generated left R-module. Then, the monomorphic image of $e + T \in I({}_R R/T)$ embeds in $M_1 \oplus M_2 \oplus \cdots \oplus M_n$, where $\{1, \cdots, n\} \subseteq A$ and e is the identity of R. Then, R/T and hence its essentail extension $I(R/T)$ is embedded in $M_1 \oplus M_2 \oplus \cdots \oplus M_n$. It folloews that $I(R/T)$ is τ-finitely generated. Thus, we have R is τ-Artinian. Therefore, by [9, Proposition 2] there exists a representative system $\{V_1, \cdots, V_n\}$ of (non-isomorphic) τ-cocritical left R-module which are modules of quotient of themselves with respect to τ such that $I(V_1) \oplus \cdots \oplus I(V_n)(= E$, say) is a cogenerator of the class of τ-torsion free modules. There exists a monomorphism $f: I(V_1) \rightarrow \oplus_{\alpha \in A} M_\alpha$, where M_α is a τ-finitely generated left R-module. We can select an $\alpha \in A$ so that the canonical injection $f_\alpha: I(V_1) \rightarrow M_\alpha$ satisfies $(V_1) f_\alpha \neq 0$. Since V_1 is τ-cocritical and $(V_1) f_\alpha$ is τ-torsion free, we can deduce that $ker f_\alpha \cap I(V_1) = 0$. However, $I(V_1)$ is a uniform left R-module and hence it is embedded in the τ-finitely generated module M_α. Thus, E is τ-finitely generated.

(ii) $\Rightarrow$ (i). Let M be a τ-torsion free injective left R-module. Since R is τ-Artinian, M is a direct sum of injective indecomposable submodules each of which is an injective hull of a τ-cocritical left R-modules (cf. [10], [13]). Therefore, from the same argument as above implies that M is embedded in a direct sum of copies of E.

Now, in the following we denote by U a left R-module and $T = \mathrm{End}\,({}_R U)$. Further, let us denote by τ_1 (resp. τ_2) the hereditary torsion theory of the ring R (resp. T) cogenerated by the injective hull $I({}_R U)$ of the left R-module U (resp. $I(U_T)$ of the left R-module ${}_R U$ (resp. right T-module U_T). For a left R-module (resp. right T-module) M we denote by $Q_{\tau_1}(M)$ (resp. $Q_{\tau_2}(M)$) the module of quotient with respect to τ_1 (resp. τ_2). We shall say that M is a τ_1-quotient module, if $Q_{\tau_1}(M) = M$. Further, we shall say that a left

R-module M is U-torsionless, if M is embedded in a direct product of copies of U.

Lemma 5. *Let U be a left R-module such that* $T = End\,({}_RU)$ *and every cyclic T-submodule of* $I(U_T)$ *is U-torsionless. Then, an R-submodule K of* $U^{(1)}\oplus U^{(2)}\oplus\cdots\oplus U^{(n)}$, *a finite direct sum of copies of U, is isomorphic U, if and only if there exists a finite subset* $F = \{t_1, t_2, \cdots, t_n\}$ *of T such that the left annihilator of F in* $I(U_T)$ *is* $\{0\}$ *and* $K = \{((x)t_1, (x)t_2, \cdots, (x)t_n) \mid x \in U\}$.

Proof. Assume that K is the image of an R-monomorphism $f: U \to U^{(1)}\oplus U^{(2)}\oplus\cdots\oplus U^{(n)}$. Let $t_i: U \to U^{(i)}$ be the canonical mapping, $i=1, \cdots, n$. Then, $K = \{((x)t_1, (x)t_2, \cdots, (x)t_n) \mid x \in U\}$. Since f is a monomorphism, the left annihiulator of F in U is $\{0\}$, where $F=\{t_1, t_2, \cdots, t_n\}$. Let $y \in I(U_T)$ and $yF = 0$. Since yT is τ_2-torsion free, there exists a T-monomorphism $g: yT \to \Pi_{b\in B} U^{(b)}$, where $U^{(b)}$ is a copy of U and B is an index set. Put $g(y) = \{u_b\}_{b\in B}$. Then, it is evident that $u_bF = 0$ and hence $u_b = 0$, $b \in B$. Consequently, $y = 0$ and this holds the "*only if*" part.

Next, we prove the "*if*" part. Since the left annihilaror of F in U is $\{0\}$, it is easily checked that the R-homomorpfism $f: U \to U^{(1)}\oplus U^{(2)}\oplus\cdots\oplus U^{(n)}$ defined by $(x)f = ((x)\,t_1, (x)\,t_2, \cdots, (x)\,t_n)$, $x \in U$, is a monomorphiam. Hence the consequence is imeidiate.

In view of Lemma 5 and [8, Theorem1] we have the following result.

Proposition 6. Let U be a left R-module such that $T = \mathrm{End}\,({}_RU)$ is τ_2-Noetherian and every cyclic T-submodule of $I(U_T)$ is U-torsionless. Then, $Q_{\tau_2}(U) = U$, if and only if, $U^{(1)}\oplus U^{(2)}\oplus\cdots\oplus U^{(n)}/K$ is U-torsionless for every R-submodule K of $U^{(1)}\oplus U^{(2)}\oplus\cdots\oplus U^{(n)}$ such that K is isomorphic to U, where $U^{(1)}\oplus U^{(2)}\oplus\cdots\oplus U^{(n)}$ is an arbitrary finite direct sum of copies of the right R-module U.

Theorem 7. Let *U be a faithful left R-module and* $T = \mathrm{End}({}_RU)$. *Then, the following conditions are equivalent.*

(i) *U is a* τ_1*-fnitely generated left R-module and every* τ_1*-torsion free left R-module is embedded in a direct sum of copies of U and* $U^{(1)}\oplus U^{(2)}\oplus\cdots\oplus U^{(n)}/K$ *is U-torsionless for every R-submodule K of* $U^{(1)}\oplus U^{(2)}\oplus\cdots\oplus U^{(n)}$ *such that K is isomorphic to U, where* $U^{(1)}\oplus U^{(2)}\oplus\cdots\oplus U^{(n)}$ *is an arbitrary finite direct sum of copies of U.*

(ii) *R is* τ_1*-Artinian,* $Q_{\tau_1}(U) = Q_{\tau_2}(U)$ *and there exists a duality between the class* C_1 *of* τ_1*-finitely generated* τ_1*-quotient modules and the class* C_2 *of* τ_2*-finitely generated* τ_2*-quotient modules via functors*

$$\mathrm{Hom}\,(-, {}_RU): C_1 \to C_2 \text{ and } \mathrm{Hom}\,(-, U_T): C_2 \to C_1.$$

Proof. (i) ⇒ (ii). By Theorem 4 R is τ_1-Artinian. Therefore, from [6, Therem 2.1] every cyclic submodule of the right T-module $I(U_T)$ is U-torsionless and T is τ_2-Artinian. Then, T is τ_2-Noetherian by a result of [10] and hence we have $Q_{\tau_2}(U) = U$ from Proposition 6. It follows that $Q_{\tau_1}(U) = U = Q_{\tau_2}(U)$ hence (ii) holds by [6, Theorem 2. 3].

(ii) ⇒ (i). From [6, Theorem 2.3] every finitely generated τ_1-torsion free left R-module is U-torsionless, since $U = Q_{\tau_1}(U) = Q_{\tau_2}(U)$. Hence by [6, Lemma 1. 2] every τ_1-torsion free left R-module is U-torsionless. Since R is τ_1-Artinian from the same argument as the proof of Theorem 4 we have that every τ_1-torsion free left R-module is embedded in a direct sum of copie of U. This holds (i) from [6. Thearem 2.1]

Faith-Walker proved in [2] that a ring R is quasi-Frobenius, if and only if for every injective left R-module M there exists a set of primitive idempotent elements $\{e_i\}_{i\in I}$ such that M is isomorphic to $\oplus_{i\in I} Re_i$. In view of [5] and [8] generalizing this result we can easily see that ring a R is a semiprimary

QF-3 maximal (two-sided) quotient ring of itself, if and only if for every $I(R)$-torsionless injective left R-module M there exists a set of primitive idempotent elements $\{e_i\}_{i \in I}$ such that M is isomorphic to $\oplus_{i \in I} Re_i$ and $R^{(1)} \oplus \cdots \oplus R^{(n)}/K$ is torsionless for every R-submodule K of $R^{(1)} \oplus \cdots \oplus R^{(n)}$ such that K is isomorphic to R, where $R^{(1)} \oplus \cdots \oplus R^{(n)}$ is an arbitrary finite direct sum of copies of the left R-module R.

Proposition 8. Let τ (resp. τ') be the hereditary torsion theory cogenerated by $I({}_RR)$ (resp. $I(R_R)$). If every τ-torsion free cyclic left R-module and every τ'-torsion free cyclic right R-module are embedded in free R-modules, then every τ-closed left ideal (resp. τ'-closed right ideal) is τ-finitely generated (resp. τ'-finitely generated).

Proof. For every subset X of R we denote by $\gamma(X)$ (resp. $l(X)$) the right (resp. left) annihilator of X in R. Let L be a τ-closed left ideal and $K = \gamma(L)$. Then, K is a τ'-closed right ideal and hence from the hypothesis the right R-module R/K is embedded in a finite direct sum of copies of (the right R-module) R. Hence there exists a finite subset $F = \{r_1, r_2, \cdots, r_n\}$ of R such that $\gamma(F) = K$. On the other hand, L is an annihilator left ideal, since R/L is a torsionless left R-module. It follows that $L = l(\gamma(F))$. Let P be the right annihilator of F in $I({}_RR)$ and V the left annihilator of P in R. It is not hard to see that $V = L$. On the other hand, since $Rr_1 + Rr_2 + \cdots + Rr_n$ is a τ-dense submodule of V, the consequence is immediate.

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