Bulletin of Tokyo Gakugei University, Sect. IV, Vol. **53** pp. 19～23, 2001

Direct Product Decomposition of Semi-Hereditary Rings

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(*Received for publication; March 21, 2001*)

Masaike, K., and Ozaki, Y.: Direct Product Decomposition of Semi-Hereditary Rings. Bull. Tokyo Gakugei Univ. Sect. **4**, 53: 19-23(2001) ISSN 0371-6813

Abstract

Let R be a right semi-hereditary ring with a maximal right quotient ring Q such that Q is flat as a right module and the canonical mapping $Q \otimes_R Q \to Q$ is an isomorphisn. If Q is a direct product of right full liner rings, then R is a direct product of rings whose maximal right quotient rings are right full linear rings.

Key words : semi-hereditary rings, non-singular rings, flat epimorphism, maximal quotient rings, full linear rings

Throughout this paper every ring R has an identity and every homomorphism between modules will be written on the opposite side of scalars. In [6] Levy proved that if R is a right Noetherian right hereditary ring, then R is a finite direct sum of rings $\{R_i \mid i = 1, 2, ..., n\}$ all of which have simple Artinian classical right quotient rings. However, hereditary Noetherian rings are not necessarily contained in classical quotient rings. Therefore, in this paper we investigate a direct product decomposition of certain right semi-hereditary rings whose maximal right quotient rings (in stead of classical right quotient ring) become a direct product of full linear rings.

Let Q be an extension ring of a ring R. Following Morita [7] we shall say that the inclusion mapping $R \to Q$ is a *left flat epimorphism*, if Q is flat as a right R-module and the canonical mapping $Q \otimes_R Q \to Q$ is an isomorphism. In [7] it is proved that the inclusion mapping $R \to Q$ is a left flat epimorphism, if and only if $QC = Q$ for every element $q \in Q$ and the left ideal $C = \{x \in R \mid xq \in R\}$. On the other hand, Cateforis has proved in [1] that if Q is a maximal right quotient ring of a right semi-hereditary ring R, then the following conditions are equivalent;

(i) Every finitely generated non-singular right R-module is projective.

(ii) R is a right non-singular ring and the inclusion mapping of R into its maximal right quotient ring is a left flat epimorphism.

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It is well known that every right non-singular ring has a right self-injective regular maximal right quotient ring. Every right semi-hereditary ring is right non-singular and every finitely generated submodule of free right R-module over a right semi-hereditary ring is projective.

Lemma 1.

Let R be a right semi-hereditary ring with a maximal right quotient ring Q such that the inclusion mapping of R into Q is a left flat epimorphism, then every annihilator right ideal of R is a direct summand of R.

Proof.

Let A be an annihilator right ideal of R. There exists a subset T of R such that $A = \{r \in R \mid Tr = 0\}$. Define an R-homomorphism α from R to $\Pi_{t \in T} R^{(t)}$, where $R^{(t)}$ is a copy of the right R-module R, by $\alpha(x) = \{tx\}_{t \in T}, x \in R$. Since Im α is a cyclic right non-singular right module over a right semi-hereditary ring R and the inclusion mapping from R to Q is the left flat epimorphism, Im α is a projective right R-module. On the other hand, Ker $\alpha = A$ and hence A is a direct summand of R.

Proposition 2.

Let R and Q be the same as in Lemma 1. Assume Q is a direct product of rings Q_i, $i \in I$, i.e., $Q = \Pi_{i \in I} Q_i$. Let S_i be the projection of R in Q_i. Then, $S_i = Q_i \cap R$, $i \in I$, and Q_i is a maximal right quotient ring of a right semi-hereditary ring R_i, for every $i \in I$.

Proof.

It is evident that for each $j \in I$, $Q_j = \{q \in Q \mid (\Pi_{i \neq j} Q_i)q = 0\}$. Put $R_j = Q_j \cap R$. Then R_j is a right annihilator of $\Pi_{i \neq j} Q_j$ in R. Let y be an element of $\Pi_{i \neq j} Q_j$. Since yR is a nonsingular cyclic right R-module and hence projective, there exists an R-monomorphism $f: yR \to R^{(1)} \oplus R^{(2)} \oplus \cdots\cdots \oplus R^{(n)}$ a finite direct sum of copies of the right R-module R. Put $f(y) = (r_1, r_2, \cdots, r_n)$. Since yR is isomorphic to Im f, it is easily checked that for every a in R, $ya = 0$ if and only if $r_i a = 0$, $i = 1, \cdots, n$. This implies that R_j is an annihilator right ideal of a subset of R. From Lemma 1 it follows that $R = R_j \oplus K$, where K is a right ideal of R. Let r be an element of R. Then, we can write $r = r_j + k$, where $r_j \in R_j$, $k \in K$. Suppose $kQ_j \neq 0$. Let p be an element in Q_j such that $kp \neq 0$. Then, we can select $a \in R$ so that $pa \in R_j$ and $0 \neq kpa \in R_j \cap$ K, since R_j is an ideal. This is a contradiction and hence we have that $kQ_j = 0$. Put $\pi_i(k) = q_i$ $i \in I$, where π_i is the canonical projection from Q to Q_i. Then, $0 = kQ_j$ implies $q_j Q_j = 0$, i.e., $q_j = 0$. Therefore, $\pi_j(r) = r_j$ and hence $S_j = R_j$.

Let e be the identity of R. If we put $e = \{e_i\}_{i \in I}$, e_i is the identity of R_i, which is a subring of Q_i. It is obvious that Q_i is a right self-injective and regular ring. On the other hand, it is easy to see that R_i is a right semi-hereditary and hence right non-singular ring. Furthermore, Q_i is an essential extension of R_i as a right R_i-module and hence Q_i is a maximal right quotient ring of R_i.

A ring R is said to be a *right full linear ring*, if R is a ring of all linear transformations of a right vector space over a division ring. A submodule M of a right module N over a ring R is called a *closed submodule*, if M has no proper essential extension in N. In [3] it is proved that a right non-singular ring R has a maximal right quotient ring which is a direct product of a right full linear ring, if and only if every non-zero closed right ideal contains a minimal closed right ideal.

Now, we are able to prove the following

Theorem 3.

Let R be a right semi-hereditary ring with a maximal right quotient ring Q. Then, the following *conditions are equivalent;*

(i) *Every non-zero closed right ideal contains a minimal closed right ideal and the inclusion mapping* $R \to Q$ *is a left flat epimorphism.*

(ii) Q *is a direct product of right full linear rings and the inclusion mapping* $R \to Q$ *is a left flat epimorphiam.*

(iii) R *has a direct product decomposition* $R = \Pi_{i \in I} R_i$ *such that every* R_i *has a maximal right quotient ring* Q_i *which is a right full linear ring such that the inclusion mapping* $R_i \to Q_i$ *is a left flat epimorphism for every* $i \in I$.

Proof.

Since R is right non-singular, the equivalence of (i) and (ii) is evident.

(ii) $\Rightarrow$ (iii). Put $Q = \Pi_{i \in I} Q_i$, where Q_i is a right full linear ring. Then, from Proposition 2 the projection of R to Q_i is R_i and $R = \Pi_{i \in I} R_i$ such that every R_i has a maximal right quotient ring Q_i which is a right full linear ring, where $R_i = Q_i \cap R$. Let q_j be an element of $Q_j, j \in I$. Put $A = \{r \in R \mid rq_j \in R\}$. Let a be an element of A denoted by $\{a_i\}_{i \in I}$. It is evident that $aq_j = a_j q_j \in R_j$ and hence $a_j \in R_j \cap A$. Thus, we can deduce that $A = \Pi_{i \neq j} R_j \oplus R_j \cap A$. Since the inclusion mapping $R \to Q$ is a left flat epimorphim, $QA = Q$ and hence $Q\,(R_j \cap A) = Q_j$. It follows that

$$Q = \Pi_{i \in I} Q_i = Q\,(\Pi_{i \neq j} R_j) \oplus Q\,(R_j \cap A) = \Pi_{i \neq j} Q_i \oplus Q_j\,(R_j \cap A).$$

Therefore, we have $Q_j\,(R_j \cap A) = Q_j$ and this implies that the inclusion mapping from the ring R_j to the ring Q_j is a left flat epimorphism and this completes the proof.

(iii) $\Rightarrow$ (ii). $\Pi_{i \in I} Q_i$ is a maximal right quotient ring of the right semi-hereditary ring $\Pi_{i \in I} R_i$, i.e., $Q = \Pi_{i \in I} Q_i$. Let $q = \{q_i\}_{i \in I}$ be an element of Q. Put $B = \{x \in R \mid xq \in R\}$ and $B_i = \{x \in R_i \mid xq_i \in R_i\}$. Since the inclusion mapping $R_i \to Q_i$ is a left flat epimorphism for every $i \in I$, $Q_i B_i = Q_i$. Further, we can check $B = \Pi_{i \in I} B_i$ and then,

$$QB = (\Pi_{i \in I} Q_i)(\Pi_{i \in I} B_i) = \Pi_{i \in I} Q_i B_i = Q,$$

so that the inclusion mapping R $\to$ Q is a left flat epimorphiam.

A ring R is called *right Kasch ring*, if every proper right ideal has non-zero left annihilator. Assume that a ring R has a right Kasch maximal right quotient ring Q and q is an element of Q. Put $C = \{x \in R \mid qx \in R\}$. Since C is a dense right ideal of R, the left annihilator of C in Q is $\{0\}$ and hence the left annihilator in Q of CQ is $\{0\}$, too. Therefore, $CQ = Q$ and then, the inclusion mapping $R \to Q$ is a right flat epimorphim. A right and left Kasch ring is said to be a *Kasch ring*. It is obvious that every semi-simple Artinian ring is a Kasch ring.

A submodule M of a right module N over a ring R is said to be *dense submodule* (see [5]), if $Hom_R\,(N/M, E\,(R_R)) = 0$, where $E\,(R_R)$ is the injective hull of the right R-modle R. If M is a dense right R-submodule in Q, where Q is a maximal right quotient ring of R, the left annihilator of M in Q is $\{0\}$. If Q is a maximal right quotient ring and also a maximal left quotient ring of R, in the following we shall say that Q is a *maximal two-sided quotient ring* of R.

Lemma 4.

Let R *be a ring with a maximal right quotient ring* Q *which has a direct product decomposition* $Q = \Pi_{i \in I} Q_i$ *such that each* Q_i *is a right Kasch ring,* $i \in I$. *Then,*

(i) *The inclusion mapping* $R \to Q$ *is a right flat epimorphiam.*

(ii) *Assume each* Q_i *is a Kasch ring. Then,* Q *becomes a maximal two-sided quotient ring of* R, *if and only if the inclusion mapping* $R \to Q$ *is a left flat epimorphiam.*

Proof.

(i). Let q be an element of Q such that $q = \{q_i\}_{i \in I}$. Let $j \in I$ be arbitrary. Set $C = \{x \in R \mid qx \in R\}$

and $C_j = \{x \in R_j \; q_j x \in R_j\}$. Since $R_j = Q_j \cap R$ is a dense right R-submodule of Q_j and C is a dense right R-submoule of R, $R_j \cap C$ is a dense right R-submodule of Q_j. Therefore, $\Pi_{i \neq j} Q_j \oplus (R_j \cap C)$ is a dense right R-submodule of $Q = \Pi_{i \in I} Q_i$. It follows that the left annihilator of $R_j \cap C$ in Q_j is $\{0\}$, so that $(R_j \cap C) Q_j = Q_j$, since Q_j is a right Kasch ring. Hence $Q = (\Pi_{i \in I} (R_j \cap C))Q \subseteq CQ$, i.e., $CQ = Q$ and this implies the inclusion mapping $R \rightarrow Q$ is a right flat epimorphiam.

(ii). Assume Q is a maximal two-sided quotient ring of R. From the left-right symmetry of (i) the inclusion mapping $R \rightarrow Q$ is a left flat epimorphiam.

Conversely, assume the inclusion mapping $R \rightarrow Q$ is a left flat epimorphiam. Then, Q is contained in a maximal left quotient ring of R. On the other hand, since each Q_i is a Kasch ring, so is Q. Hence Q is a maximal left quotient ring of itself. Thus, Q becomes a maximal two-side quotient ring of R.

Let R be a right full linear ring such that R is a endomorphism ring of a right vector space V over a division ring D. Then, R is left self-injective, if and only if V is finite dimensional over D (cf. [2, Proposition 2. 23]).

If R is a right semi-hereditary ring with a maximal right quotient ring Q such that the inclusion mapping $R \rightarrow Q$ is a left flat epimorphiam, then R becomes a left semi-hereditary ring (see [2. Theorem 5.18]). From these results we have the following

Theorem 5.

Assume R is a ring with a maximal two-side quotient ring. Then, the following conditions are equivalent;

(i) *R is right semi-hereditary and every non-zero closed right ideal contains a minimal closed right ideal.*

(ii) *R is left semi-hereditary and every non-zero close left ideal contains a minimal closed left ideal.*

(iii) *R has a direct product decomposition $R = \Pi_{i \in I} R_i$, such that each R_i is a right and left semi-hereditary ring with a simple Artinian maximal two-sided quotient ring.*

Proof.

(i) $\Rightarrow$ (iii). Clearly, Q is a maximal left quotient ring of itself and hence self-injective on left side, since Q is a regular ring. Therefore, each Q_i is a left self-injective ring, which is isomorphic to a ring of all linear transformations of a finite dimensional vector space, i.e., Q_i is a simple Artinian ring and hence a Kasch ring. It follows that the inclusion mapping $R \rightarrow Q$ is a left flat epimorphiam from Lemma 4 (ii). Then, R is left semi-hereditary and R has a direct product decomposition $R = \Pi_{i \in I} R_i$ such that every R_i is right and left semi-hereditary which has a maximal two-sided quotient ring Q_i.

(iii)$\Rightarrow$ (i). This is obvious from Theorem 3.

A right R-module M is said to be *finite Goldie dimensional*, if M satisfies the ascending chain condition (and hence descending chain condition) on closed submodules. It is well known that a ring R has a semi-simple Artinian maximal right quotient ring, if and only if R is right non-singular and finite Goldie dimensional as a right R-module.

Corollary 6.

Let R be a right semi-hereditary ring with a maximal two-sided quotient ring. If R is finite Goldie dimensional as a right R-module, then R is a finite direct sum of right and left semi-hereditary rings which have simple Artinian maximal two-sided quotient rings.

If R is a semi-prime right Goldie ring with a classical right quotient ring $Q = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_n$, where each Q_i is a simple Artinian ring. Then, $Q_i \cap R$, whose right annihilator and left annihilator (in R) coincide, is a minimal right annihilator and hence a minimal left annihilator (see [6]). Now, we consider the case where R has not necessarily a classical right quotient ring. Set $\Omega = \{$two-side ideal $A \neq 0$ of $R \mid A$ is a right annihilator and a left annihilator such that the right annihilator of A and a left annihilator of A coincide$\}$.

Proposition 7.

Let R be a ring with a semi-simple Artinian maximal two-sided quotient ring $Q = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_n$, where each Q_i is a simple Artinian ring. Then every $Q_i \cap R = R_i$ is a minimal element of Ω.

Proof.

It is easy to see that R_1 is a right (and left) annihilator of $(Q_2 \oplus \cdots\cdots \oplus Q_n) \cap R$ and contained in Ω. Assume A is a minimal element of Ω and A is contained in R_1. Suppose $R_1 \neq A$. Let B be the right annihilator of A. Then, $B \supsetneqq (Q_2 \oplus \cdots\cdots \oplus Q_n) \cap R$ and this implies $B \cap Q_1 \neq 0$.

Assume, at first, $A \cap B = 0$. Since AQ_1 and BQ_1 are R-submodules of the left R-module Q, which is a maximal left quotient ring of R, and $AQ_1 \cap BQ_1 = 0$, we have $Q_1AQ_1 \cap Q_1BQ_1 = 0$. This is a contradiction since Q_1 is a simple ring.

Next, assume $A \cap B \neq 0$. Clearly, A and B are contained in Ω and hence so $A \cap B$. Since A is minimal in Ω, it follows that $A \subseteq B$ and then, $A^2 = 0$. Let C be a non-zero left ideal of R. If $AC = 0$, we have $C \subseteq B$ and hence $B \cap C \neq 0$. On the other hand, assume $AC \neq 0$. Since $A(AC) = 0$, i.e., $AC \subseteq B$ and hence $B \cap C \supseteq A \cap C \supseteq AC \neq 0$. This implies B is an essential left of R. Since R is a left non-singular ring, B has no non-zero right annihilator in R. However, this is a contradiction, as B is a left annihilator of A. This completes the proof.

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